

# Arbitrary order Hessenberg–quasiseparable matrices and polynomials

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# Orthogonal Polynomials Related to Structured Matrices

## Moment Matrices

➡ **Hankel matrices.** Defined by  $\mathcal{O}(n)$  parameters  $\{h_k\}$ .

$$H = \left[ h_{k+j} \right] = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_1 & h_2 & \ddots & & \vdots \\ h_2 & & \ddots & & h_{2n-3} \\ \vdots & \ddots & & h_{2n-3} & h_{2n-2} \\ h_{n-1} & \cdots & h_{2n-3} & h_{2n-2} & h_{2n-1} \end{bmatrix}$$

➡ **Toeplitz matrices.** Defined by  $\mathcal{O}(n)$  parameters  $\{t_k\}$ .

$$C = \left[ t_{k-j} \right] = \begin{bmatrix} t_0 & t_{-1} & \cdots & \cdots & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & t_0 & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{bmatrix}$$

# Orthogonal Polynomials Related to Structured Matrices

## Moment Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- For a given inner product, the **moment matrix** is

$$M = [\langle x^k, x^j \rangle] = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle & \dots & \langle 1, x^n \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \langle x, x^2 \rangle & \dots & \langle x, x^n \rangle \\ \langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle & \dots & \langle x^2, x^n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \langle x^n, 1 \rangle & \langle x^n, x \rangle & \langle x^n, x^2 \rangle & \dots & \langle x^n, x^n \rangle \end{bmatrix}$$

- For an inner product defined by integration on the **real line**,

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)w^2(x)dx, \quad \Rightarrow \quad \langle x^k, x^j \rangle = \int_a^b x^{(k+j)}w^2(x)dx,$$

and  $M$  is **Hankel**.

- Hankel** matrices are related to **real-orthogonal polynomials**.

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- For an inner product defined by integration on the **unit circle**,

$$\langle p(x), q(x) \rangle = \int_{-\pi}^{\pi} p(e^{i\theta}) \cdot \overline{q(e^{i\theta})} w^2(\theta) d\theta \Rightarrow \langle x^k, x^j \rangle = \int_{-\pi}^{\pi} x^{(k-j)} w^2(\theta) d\theta,$$

and  $M$  is **Toeplitz**.

- Toeplitz** matrices are related to **Szegő polynomials**.

# Orthogonal Polynomials Related to Structured Matrices

## Recurrent Matrices

➡ **Tridiagonal matrices.** Defined by  $\mathcal{O}(n)$  parameters.

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

➡ **Unitary Hessenberg matrices.** Defined by  $\mathcal{O}(n)$  parameters.

$$U = \begin{bmatrix} -\rho_1 \rho_0^* & -\rho_2 \mu_1 \rho_0^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_1 \rho_0^* \\ \mu_1 & -\rho_2 \rho_1^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_2 \rho_1^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} \rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n \rho_{n-1}^* \end{bmatrix}$$

## Orthogonal Polynomials Related to Structured Matrices

### Recurrent Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- The system of polynomials defined by  $\mathbf{r}_k(x) = \det(xI - T)_{(k \times k)}$  where

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

are real-orthogonal polynomials.

# Orthogonal Polynomials Related to Structured Matrices

## Recurrent Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- The system of polynomials defined by  $\mathbf{r}_k(x) = \det(xI - U)_{(k \times k)}$  where

$$U = \begin{bmatrix} -\rho_1 \rho_0^* & -\rho_2 \mu_1 \rho_0^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_1 \rho_0^* \\ \mu_1 & -\rho_2 \rho_1^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_2 \rho_1^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} \rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n \rho_{n-1}^* \end{bmatrix}$$

are Szegő polynomials.





## A Brief Interruption

### Vadim's Talk: Fiedler Matrices

▣ Also, the companion matrix

$$U = \begin{bmatrix} -c_1 & -c_2 & \cdots & -c_{n-1} & -c_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and the Fiedler matrix

$$T = \begin{bmatrix} -c_1 & -c_2 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & -c_3 & 0 & -c_4 & 1 & & \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & -c_5 & 0 & -c_6 & 1 \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \ddots \end{bmatrix}$$

are related to **Horner polynomials** via  $\mathbf{r}_k(x) = \det(xI - T)_{(k \times k)}$

## Generalizations of these Structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	????????????????????????????????

## Generalizations of these Structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	quasiseparable matrices

## Quasiseparable Matrices

- ▣ **Definition.** A matrix  $C$  is  $(H, m)$ -**quasiseparable** if it is strongly upper Hessenberg (nonzero subdiagonals, zeros below that) and

$$\max \text{Rank} C_{12} = m$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[ \begin{array}{c|c} * & C_{12} \\ \hline & * \end{array} \right]$$

- ▣ **Previous work.** Bini, Chandrasekaran, Eidelman, Fasino, Gemignani, Gohberg, Gu, Kailath, Koltracht, Mastronardi, Olshevsky, Tyrtyshnikov, Van Barel, Vandebril...
- ▣ A system of polynomials related to an  $(H, m)$ -quasiseparable matrix  $C$  as **characteristic polynomials** of principal submatrices of  $C$ , i.e.

$$r_k(x) = \det(xI - C_{k \times k})$$

will be called  $(H, m)$ -**quasiseparable polynomials**.

## Important Special Cases of Quasiseparable Matrices

### Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- The system of polynomials  $r_k(x) = \det(xI - C_{k \times k})$  associated with  $C$  are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k)r_{k-1}(x) - \frac{g_{k-1}}{q_k}r_{k-2}(x)$$

- The matrix  $C$  is  **$(H, 1)$ -quasiseparable**.

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## Important Special Cases of Quasiseparable Matrices

### Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

► The system of polynomials  $r_k(x) = \det(xI - C_{k \times k})$  associated with  $C$  are the **Szegő polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x r_{k-1}(x) \end{bmatrix}$$

► The matrix  $C$  is  $(H, 1)$ -**quasiseparable**.

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## The Difference Between These Motivating Examples

### Unitary Hessenberg matrices.

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 \end{bmatrix}$$

### Tridiagonal matrices.

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$



## The Difference Between These Motivating Examples

➡ **Unitary Hessenberg matrices.** strictly upper triangular part is part of a low rank matrix.

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 \\ \frac{-\rho_1^* \rho_1}{\mu_1} & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 \\ -\rho_2^* \rho_1 & \frac{-\rho_2^* \rho_2}{\mu_2} & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 \\ \frac{\mu_1 \mu_2}{-\rho_3^* \rho_1} & \frac{\mu_2}{-\rho_3^* \rho_2} & \frac{-\rho_3^* \rho_3}{\mu_3} & -\rho_3^* \rho_4 \\ \mu_1 \mu_2 \mu_3 & \mu_2 \mu_3 & \mu_3 & \mu_4 \end{bmatrix}$$

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 \end{bmatrix}$$

➡ **Tridiagonal matrices.** strictly upper triangular part is NOT part of a low rank matrix.

$$C = \begin{bmatrix}
 & g_1 & 0 & 0 & 0 \\
 & & g_2 & 0 & 0 \\
 & & & g_3 & 0 \\
 & & & & g_4
 \end{bmatrix}$$

## Semiseparable matrices

▣▣▣ **Definition.** A matrix  $R$  is called  $(r_L, r_U)$ -semiseparable if for some  $r_L, r_U$  we have

$$R = D + \text{tril}(R_L) + \text{triu}(R_U),$$

where  $\text{rank}R_L = r_L$ ,  $\text{rank}R_U = r_U$ , with some  $R_L, R_U$ .

▣▣▣ **Example.**  $(1, 1)$ -semiseparable:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

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$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

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$$R = D + \text{tril}(R_L) + \text{triu}(R_U),$$

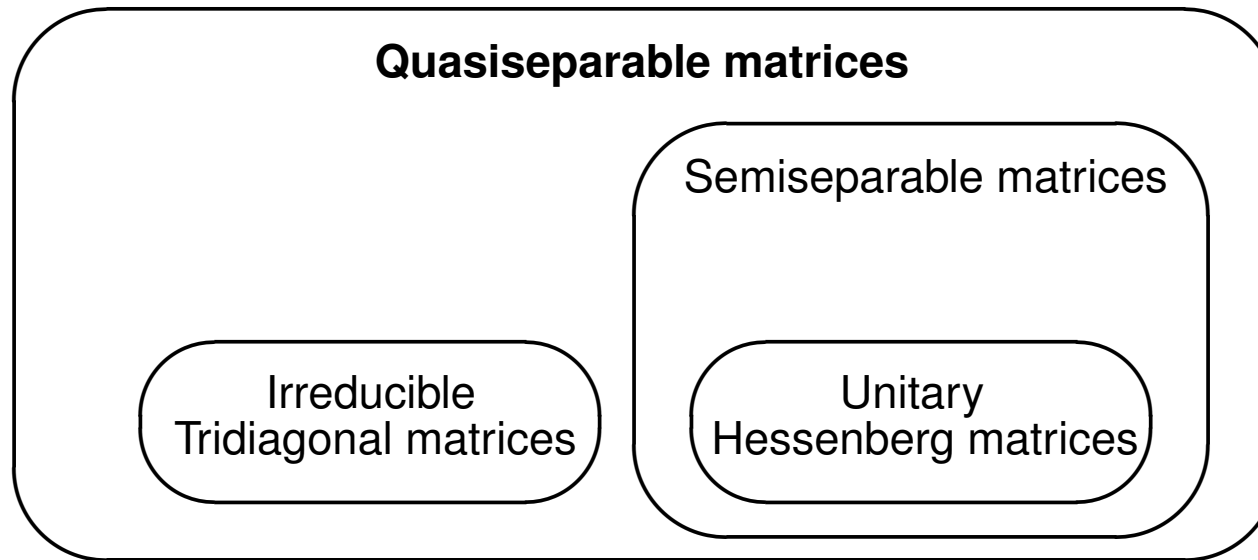
where  $\text{rank} R_L = r_L$ ,  $\text{rank} R_U = r_U$ , with some  $R_L, R_U$ .

▣ **Example.**  $(1, 1)$ -semiseparable:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

## Quasiseparable, Semiseparable, and Subclasses



## Generator Representation of an $(H, 1)$ -Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

## Generator Representation of an $(H, 1)$ -Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

► This **generator representation** exists for any  $(H, 1)$ -**quasiseparable** matrix.

# Classification of $(H, 1)$ -quasiseparable matrices in terms of recurrence relations

Joint work with Yuli Eidelman, Israel Gohberg, and Vadim Olshevsky

## Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices $A$	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

$$\mathbf{r}_k(x) = \det(xI - A)_{(\mathbf{k} \times \mathbf{k})}$$

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### Recurrence relations

$$\mathbf{r}_k(x) = \mathbf{x} \cdot \mathbf{r}_{k-1}(x)$$

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Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

### Recurrence relations

$$\mathbf{r}_k(x) = 2\mathbf{x} \cdot \mathbf{r}_{k-1}(x) - \mathbf{r}_{k-2}(x)$$

## Efficient Recurrence Relations for Quasiseparable Polynomials

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### Recurrence relations

$$\mathbf{r}_k(x) = (\alpha_k \mathbf{x} - \delta_k) \mathbf{r}_{k-1}(x) - \gamma_k \mathbf{r}_{k-2}(x)$$

## Efficient Recurrence Relations for Quasiseparable Polynomials

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Quasiseparable matrix	Quasiseparable polynomials

### Recurrence relations (2-term)

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{r}_k(x) \end{bmatrix}$$

## Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices $A$	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

### Recurrence relations (3-term)

$$\mathbf{r}_k(x) = \left( \frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) \mathbf{r}_{k-1}(x) - \left( \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) \mathbf{r}_{k-2}(x)$$

## Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices $A$	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

### Recurrence relations

????????????????????

## Three-term Recurrence Relations.

Consider the class of polynomials satisfying more general three-term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - (\beta_k x + \gamma_k)r_{k-2}(x)$$

▮▮▮ Real-orthogonal polynomials:  $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

▮▮▮ Szegő polynomials (orthogonal on the unit circle):  $\gamma_k = 0$

$$r_k(x) = \left( \frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left( \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

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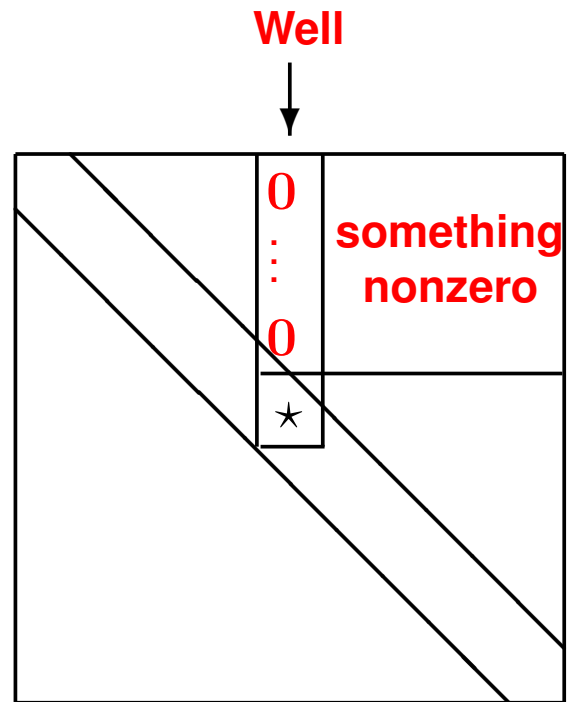
▮▮▮ Real-orthogonal polynomials:  $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

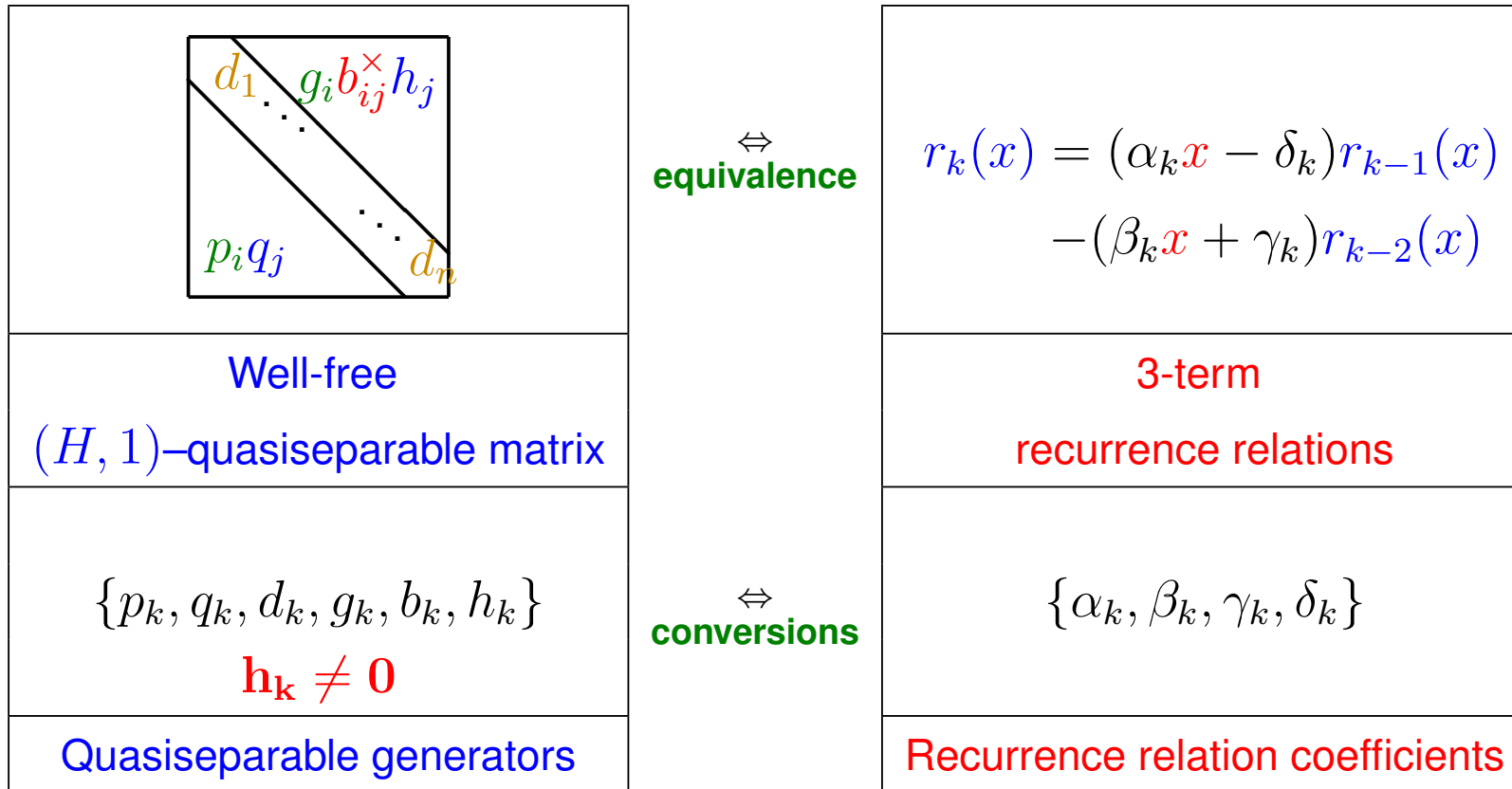
▮▮▮ Szegő polynomials (orthogonal on the unit circle):  $\gamma_k = 0$

$$r_k(x) = \left( \frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left( \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

## The Corresponding Matrix Class: Well-Free Matrices.



## Well-Free Matrices & 3-term Recurrence Relations



## Subclasses of $(H, 1)$ -Quasiseparable Matrices

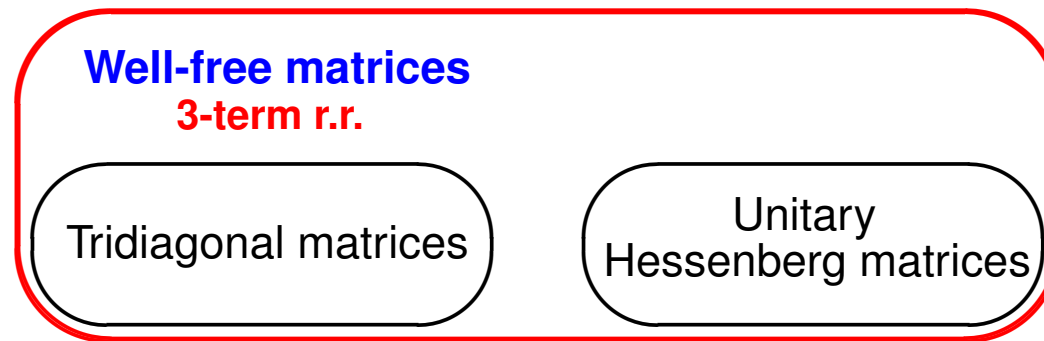
Corresponding recurrence relations

Tridiagonal matrices

Unitary  
Hessenberg matrices

## Subclasses of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations



## Szegő-type Two-term Recurrence Relations

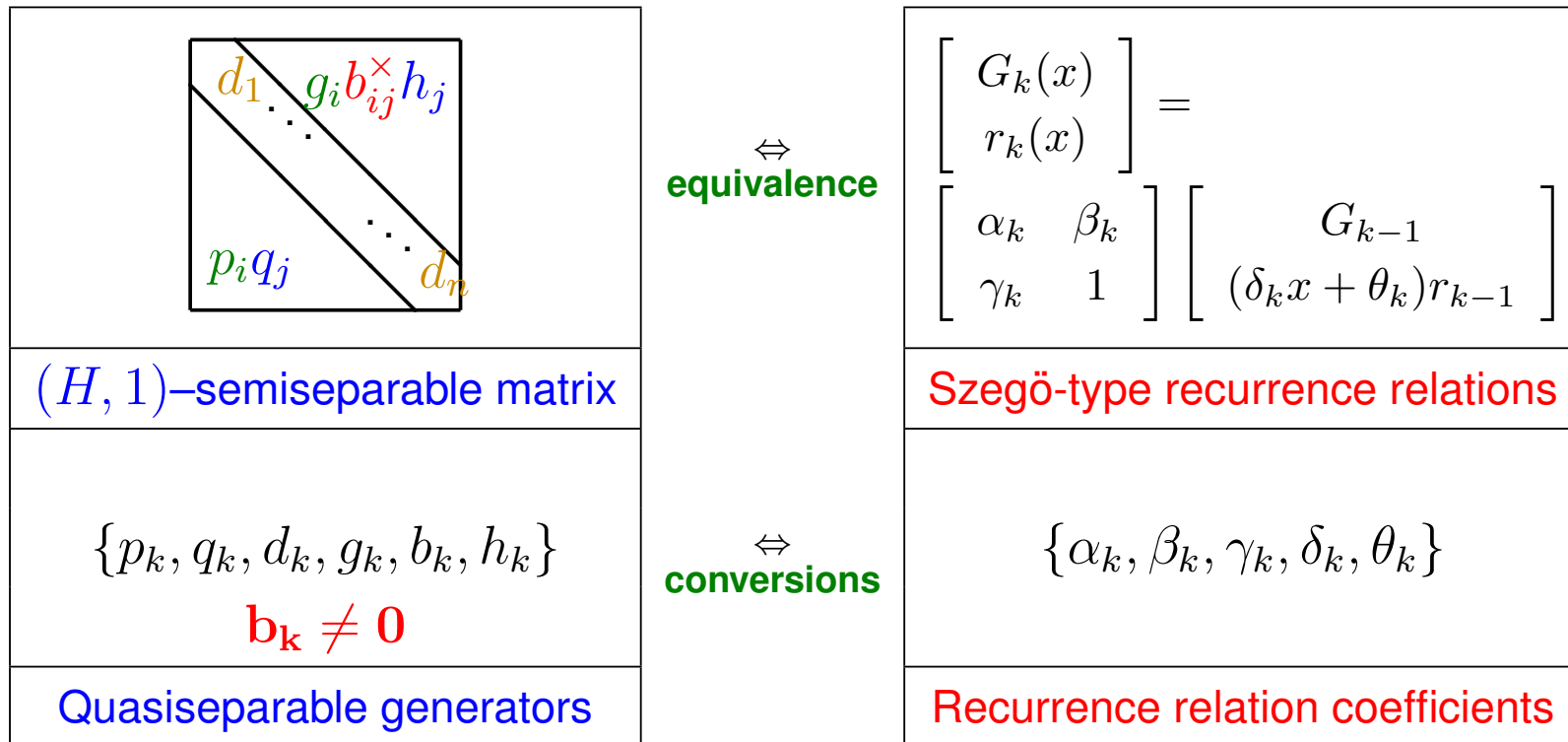
⇒ Szegő polynomials satisfy two-term recurrence relations of the form

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{xr}_k(x) \end{bmatrix}.$$

⇒ Is there a class of polynomials **larger** than Szegő that satisfy two-term recurrence relations of the form

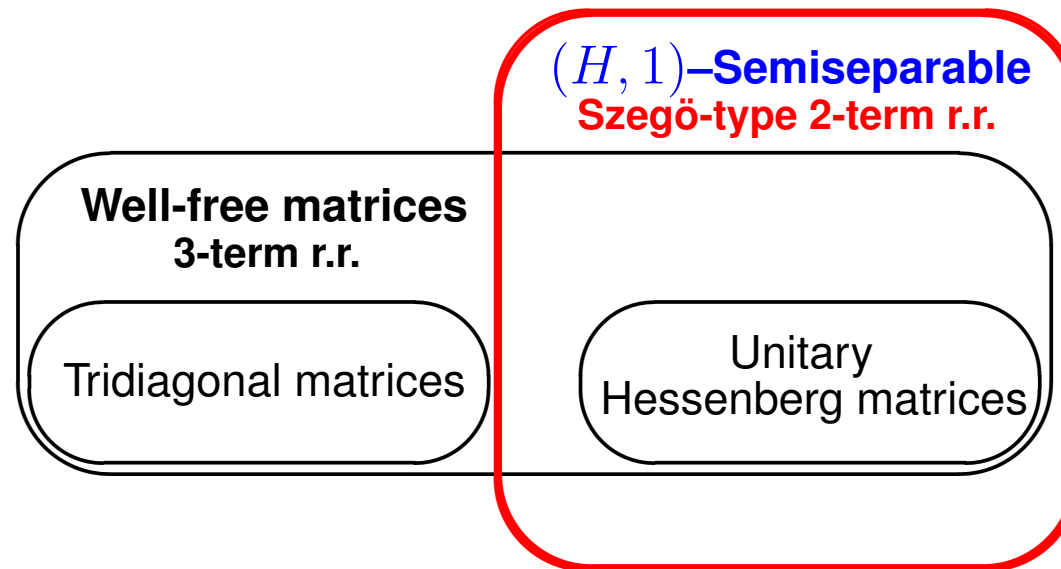
$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ (\delta_k x + \theta_k) \mathbf{r}_k(x) \end{bmatrix}?$$

## Semiseparable Matrices & Szegő-type 2-term Recurrence Relations



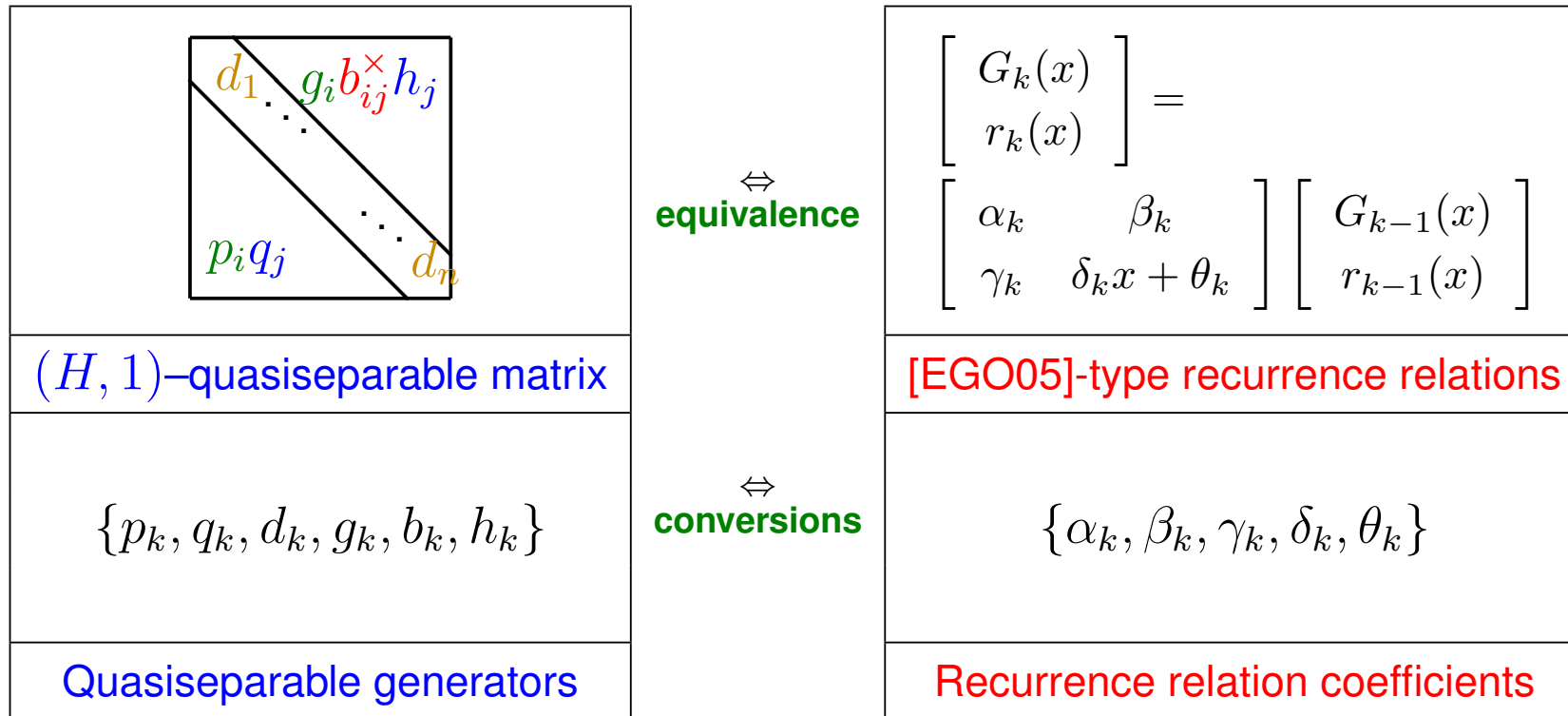
## Subclasses of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations



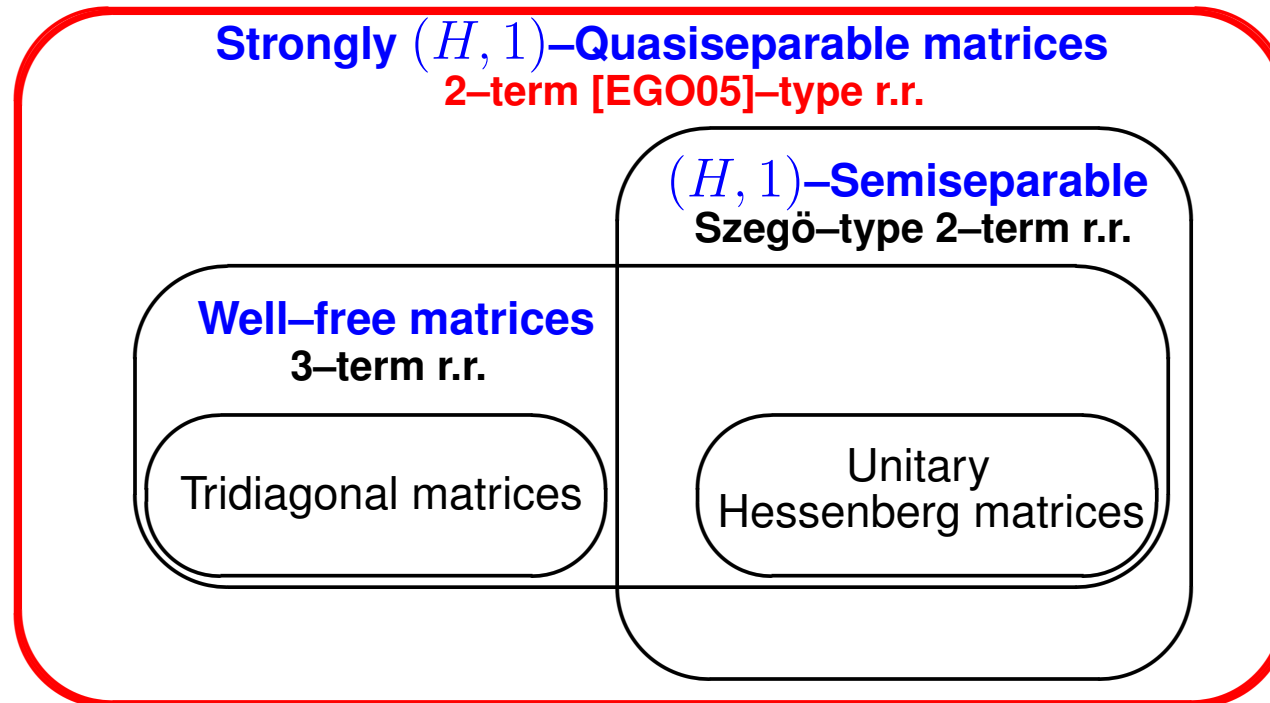
# Quasiseparable Matrices & [EGO05]-type 2-term Recurrence Relations

## A Complete Characterization of $(H, 1)$ -Quasiseparable Matrices



# Full Characterization of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations



## A Brief Interruption

### Vadim's Talk: Twisted $(H, 1)$ -Quasiseparable Matrices

- Neither CMV nor Fiedler matrices are Hessenberg, so they certainly aren't  $(H, 1)$ -quasiseparable.
- But they are **Twisted  $(H, 1)$ -quasiseparable**, by which we mean that
  - it is  $(1, 1)$ -quasiseparable, and
  - there is a  $(H, 1)$ -quasiseparable matrix whose generators can be changed into those of the twisted matrix via

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

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$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

- **Key Idea:** The recurrence relations for polynomials  $r_k(x)$  satisfied by any  $(1, 1)$ -quasiseparable matrix are.

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \lambda_k \begin{bmatrix} a_k b_k x - c_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix},$$

where  $c_k = d_k a_k b_k - q_k p_k b_k - g_k h_k a_k$ .













# Recurrence relation classification of $(H, m)$ -quasiseparable matrices

Joint work with Vadim Olshevsky and Pavel Zhlobich



## A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

## A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

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## A generator representation for $(H, m)$ -quasiseparable polynomials.

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

## A generator representation for $(H, m)$ -quasiseparable polynomials.

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$(H, 1)$ -quasiseparable generators are <b>scalars</b>	$(H, m)$ -quasiseparable generators are <b>matrices</b>
$\boxed{g_1} \times \boxed{b_2} \times \boxed{b_3} \times \boxed{h_4}$	$\boxed{g_1} \times \boxed{b_2} \times \boxed{b_3} \times \boxed{h_4}$

## What recurrence relations are satisfied by $(H, m)$ -quasiseparable polynomials?

➡ The recurrence relations satisfied by  $(H, 1)$ -quasiseparable polynomials are

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

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$$\begin{bmatrix} \boxed{F_k(x)} \\ \hline r_k(x) \end{bmatrix} = \begin{bmatrix} \boxed{\alpha_k} & \boxed{\beta_k} \\ \hline \boxed{\gamma_k} & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} \boxed{F_{k-1}(x)} \\ \hline r_{k-1}(x) \end{bmatrix}$$

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➡ The recurrence relations satisfied by  $(H, 1)$ -semiseparable polynomials are

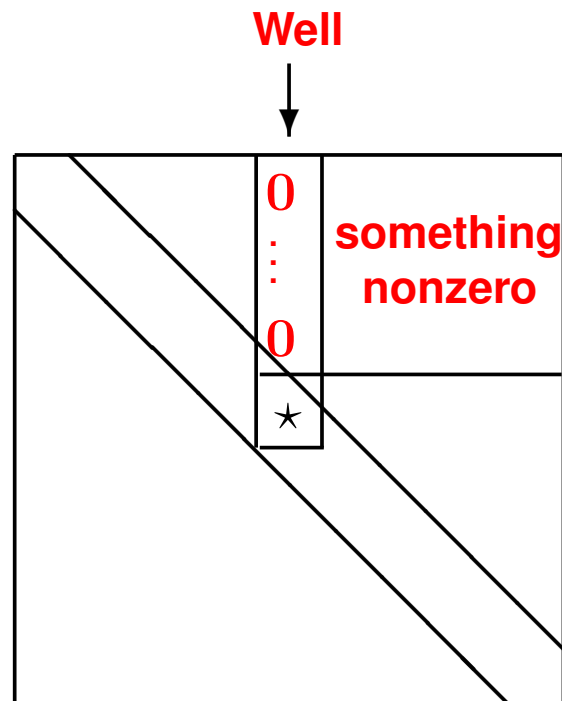
$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

➡ The recurrence relations satisfied by  $(H, m)$ -semiseparable polynomials are

$$\begin{bmatrix} \boxed{G_k(x)} \\ \hline r_k(x) \end{bmatrix} = \begin{bmatrix} \boxed{\alpha_k} & | & \boxed{\beta_k} \\ \hline \boxed{\gamma_k} & | & 1 \end{bmatrix} \begin{bmatrix} \boxed{G_{k-1}(x)} \\ \hline (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

## A generalization of well-free structure?

➡ Recall that a matrix is  $(H, 1)$ -well-free if it contains no “wells” of the form



➡ What is the **order**  $m$  version of this structure?

$(H, m)$ -well-free matrices

▣ **Definition.** A matrix is  $(H, m)$ -well-free if adding the next column to any  $m$  consecutive columns of  $C_{12}$  does **not** increase the rank.

$(H, m)$ -well-free matrices

▣▣▣ **Definition.** A matrix is  $(H, m)$ -well-free if adding the next column to any  $m$  consecutive columns of  $C_{12}$  does **not** increase the rank.

▣▣▣ **Example.**  $m = 3$ .

$$C = \left[ \begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

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▣ **Example.**  $m = 3$ .

$$C = \left[ \begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{array}{c} \text{rank } r_1 \\ \left[ \begin{array}{ccc|ccc} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{array} \right] \end{array}$$

## $(H, m)$ -well-free matrices

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rank  $r_1$ 
also rank  $r_1$

## $(H, m)$ -well-free matrices

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$\underbrace{\hspace{10em}}_{\text{rank } r_2}$

## $(H, m)$ -well-free matrices

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➡ **Example.**  $m = 3$ .

$$C = \left[ \begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

rank  $r_2$ 
also rank  $r_2$

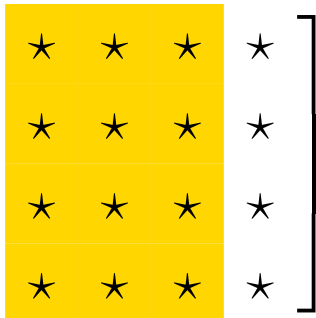
The diagram shows the matrix  $C_{12}$  with four rows and six columns. The first three columns are highlighted in yellow, and the fourth column is highlighted in a darker brown. A bracket above the first three columns is labeled "rank  $r_2$ ". A bracket below the first three columns is labeled "also rank  $r_2$ ".

## $(H, m)$ -well-free matrices

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▣ **Example.**  $m = 3$ .

$$C = \left[ \begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$



## $(H, m)$ -well-free matrices

▣ **Definition.** A matrix is  $(H, m)$ -well-free if adding the next column to any  $m$  consecutive columns of  $C_{12}$  does **not** increase the rank.

▣ **Example.**  $m = 3$ .

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rank  $r_3$ 
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also rank  $r_3$

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▶▶▶▶ **Example.**  $m = 1$ . (i.e.,  $(H, 1)$ -well-free)

$$C = \left[ \begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \end{bmatrix}$$

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## What recurrence relations are satisfied by $(H, m)$ -well-free polynomials?

▣▶ The recurrence relations satisfied by  $(H, 1)$ -well-free polynomials are

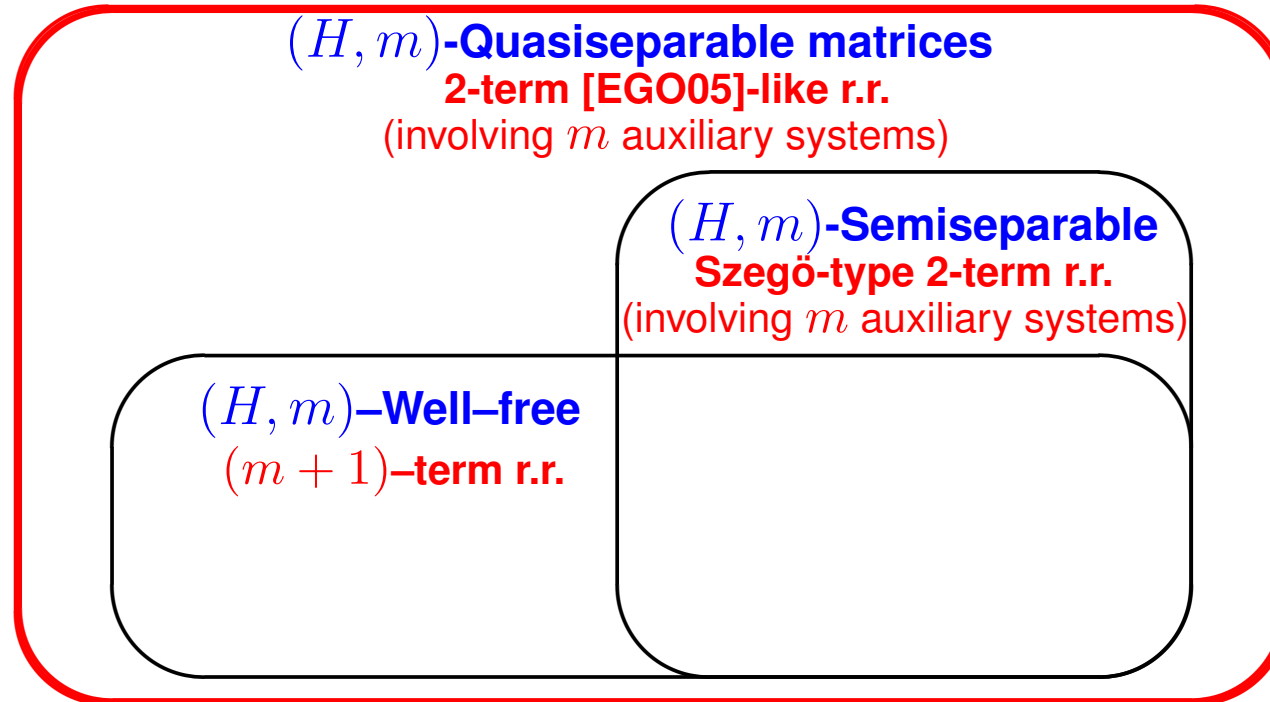
$$r_k(x) = \underbrace{(\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)}_{\text{depends on the previous two polynomials}}$$

▣▶ The recurrence relations satisfied by  $(H, m)$ -well-free polynomials are

$$r_k(x) = \underbrace{(\delta_{k,k}x + \epsilon_{k,k})r_{k-1}(x) + (\delta_{k-1,k}x + \epsilon_{k-1,k})r_{k-2}(x) + \cdots}_{\text{depends on the previous } m + 1 \text{ polynomials}}$$

## Full Characterization of $(H, m)$ -quasiseparable matrices

Corresponding recurrence relations



# Arbitrary order Hessenberg–quasiseparable matrices and polynomials

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Joint work with Yuli Eidelman, Israel Gohberg,  
Vadim Olshevsky, & Pavel Zhlobich



