Spectra of Digraphs

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LA'09, Northern Illinois University
DeKalb, August 12–14 2009

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Dedicated to Biswa Datta on his recent birthday $\#68$

Digraphs D

Digraph D: set **V** of **vertices**, and set **E** of **edges** which are ordered pairs of not necessarily distinct vertices. So loops are allowed.

An edge (u, v) $(u - - > v)$ contributes 1 to the **outdegree** of U and 1 to the **indegree** of v. A loop contributes 1 to both the indegree and outdegree of u.

So we have an **outdegree vector** $R = (r_1, r_2, \ldots, r_n)$ and **indegree vector** $S = (s_1, s_2, \cdots, s_n)$ where

$$r_1 + r_2 + \cdots + r_n = s_1 + s_2 + \cdots + s_n.$$

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Adjacency Matrix of digraph D

Order the vertices in some way: $v_1, v_2, \ldots, v_n$. The adjacency matrix is the $(0,1)$-matrix $A = [a_{ij}]$ of order n where

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise.} \end{cases} .$$

A different ordering results in the (similar) matrix PAP^T for some permutation matrix P.

In particular, the digraph is **strongly connected** iff the matrix A is **irreducible**, i.e. no P such that $PAP^t = \begin{bmatrix} A_1 & O_{r,n-r} \\ * & A_2 \end{bmatrix}$.

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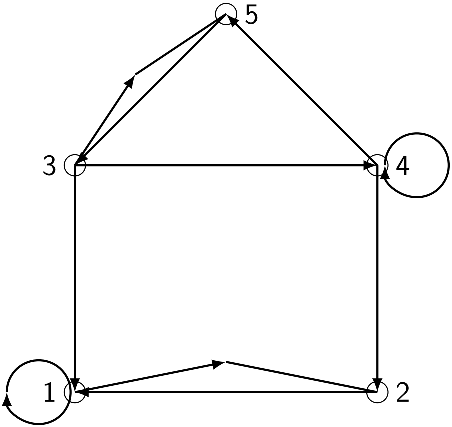
Example of a digraph D



Adjacency Matrix A of D

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Digraphs of order $n \overset{1-1}{\longleftrightarrow}$ (0,1)-matrices of order n

$$\mathcal{D}_n \overset{1-1}{\longleftrightarrow} \mathcal{A}_n$$

Outdegree vector of D is the row sum vector $(r_1, r_2, \ldots, r_n)$ of A.
Indegree vector is the column sum vector $(s_1, s_2, \ldots, s_n)$.

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Definitions

Characteristic polynomial $\chi_D(x)$ of D, **minimum polynomial** of D, **spectrum** (**eigenvalues**) $\lambda_1, \lambda_2, \ldots, \lambda_n$ of D, ... are those of its adjacency matrix A.

Unlike for symmetric matrices, the eigenvalues of D need not be real numbers. Convention is:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

The **spectral radius** of D is $\rho(D) = |\lambda_1|$.

Example $\overset{\leftrightarrow}{K}_n$ (all possible edges) $\longleftrightarrow$ J_n (all 1s matrix): eigenvalues are $n, 0, \ldots, 0$.

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Hoffman polynomial

A digraph is **regular** provided the indegree and outdegree of each vertex is some constant d. For such a D, $\rho(D) = d$.

Theorem (Hoffman/McAndrew 1965) There is a polynomial such that $p(A) = J_n$ iff D is a strongly connected, regular digraph. For such a D, the polynomial $p(x)$ of smallest degree is unique and is

$$H_D(x) = \frac{nq(x)}{q(d)}$$

where $(x - d)q(x)$ is the minimum polynomial of D. The integer d is the largest real solution of the equation $H_D(x) = n$.

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Perron-Frobenius Theory

- $\rho(D)$ is an eigenvalue,
- If D is strongly connected, there is a (unique) positive eigenvector corresponding to $\rho(D)$.
- $\min\{r_1, r_2, \ldots, r_n\} \leq \rho(D) \leq \max\{r_1, r_2, \ldots, r_n\}$. Equality on the right iff equality on the left iff D is regular.
- If k is the GCD of cycle lengths of D, then the spectrum of D is invariant under a rotation of the complex plane about the origin by the angle $2\pi/k$.

Some Definitions

- $\mathcal{D}_n(e) =:$ all digraphs with n vertices and e edges ($e \leq n^2$).
- $\mathcal{D}(e) =:$ all digraphs with e edges, number of vertices not specified..
- $\mathcal{D}_n(e \uparrow) \subseteq \mathcal{D}_n(e)$ such that the vertices can be ordered $v_1, v_2, \ldots, v_n$ so that if (v_p, v_q) is an edge, then (v_i, v_j) is an edge for all i and j with $1 \leq i \leq p, 1 \leq j \leq q$.
- $\mathcal{D}_n(e \downarrow) \subseteq \mathcal{D}_n(e)$ such that the vertices can be ordered $w_1, w_2, \ldots, w_n$ so that if (w_p, w_q) is an edge, then (w_i, w_j) is an edge for all i and j with $1 \leq i \leq p$ and $q \leq j \leq n$.
- $\mathcal{D}(e \uparrow)$ and $\mathcal{D}(e \downarrow)$ defined similarly.

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Example of $\mathcal{D}_6(25 \uparrow)$ and $\mathcal{D}_6(25 \downarrow)$

In terms of the adjacency matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

Schwarz's Theorem 1964

Theorem: The maximum (respectively, minimum) spectral radius among graphs in $\mathcal{D}(e)$ occurs among the graphs in $\mathcal{D}(e\uparrow)$ (respectively, $\mathcal{D}(e\downarrow)$).

Thus

$$\max\{\rho(D) : D \in \mathcal{D}(e)\} = \max\{\rho(D) : D \in \mathcal{D}(e\uparrow)\}$$

and

$$\min\{\rho(D) : D \in \mathcal{D}(e)\} = \min\{\rho(D) : D \in \mathcal{D}(e\downarrow)\}.$$

Similar conclusions hold with $\mathcal{D}_n(e\uparrow)$ in place of $\mathcal{D}(e\uparrow)$ and $\mathcal{D}_n(e\downarrow)$ in place of $\mathcal{D}(e\downarrow)$.

Proof uses Perron-Frobenius theory.

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Geršgorin's Theorem

Theorem: Let $R^0 = (r_1^0, r_2^0, \ldots, r_n^0)$ and $S^0 = (s_1^0, s_2^0, \ldots, s_n^0)$ be the outdegree and indegree vectors, respectively, of the digraph D^0 obtained from D by removing any loops. Then the spectrum of D is contained in the region of the complex plane defined by the union $\Gamma(D)$ of the disks

$$\{z : |z - a_{ii}| \leq r_i^0\} \quad (i = 1, 2, \ldots, n).$$

Here $A = [a_{ij}]$ is the adjacency matrix of D. Since A is a (0,1)-matrix, the disks have centers at $(0, 0)$ or $(1, 0)$ (thus only two are needed). If A has no loops, then this is no better that what one gets from the Perron-Frobenius theory.

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Generalizations of Geršgorin's Theorem

Theorem: (Same notation) The spectrum of D is contained in the region of the complex plane determined by the union of the lemniscates defined by the cycles γ of D:

$$B_\gamma(D) = \{z : \prod_{i \in \gamma} |z - a_{ii}| \le \prod_{i \in \gamma} r_i^0\}.$$

In general,

$$\min_\gamma \left\{ \left(\prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\} \le \rho(D) \le \max_\gamma \left\{ \left(\prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \right\}$$

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max $\rho(D)$ with m^2 or m^2+1 edges

(B+Hoffman, 1985)

- $\max\{\rho(D) : D \in \mathcal{D}(m^2)\} = m$, with equality iff $D = \overleftrightarrow{K_m}$.
- $\max\{\rho(D) : D \in \mathcal{D}(m^2+1)\}$, with equality iff apart from isolated vertices, (1) D is a complete digraph of order m with one additional edge; or, (2) $m = 1$ and the two edges of D join two distinct vertices in opposite directions; or, (3) $m = 2$ and apart from isolated vertices, D is obtained from the complete digraph of order 3 by removing a complete digraph of order 2.

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Bounds for max $\rho(D)$, I

Friedland (1985)

- For $1 \leq l \leq 2m$,

$$\max\{\rho(D) : D \in \mathcal{D}(m^2 + l)\} \leq \frac{m + \sqrt{m^2 + 2l}}{2}.$$

  Equality if $l = 2m$ and, apart from isolated vertices, D is obtained from $\overleftrightarrow{K}_{m+1}$ by removing a loop at one vertex.

- $\max\{\rho(D) : D \in \mathcal{D}(m^2 + 2m - 3)\} \leq \frac{m-1+\sqrt{m^2+6m-7}}{2}$. For $m \geq 3$, equality holds if and only if D is obtained from $\overleftrightarrow{K}_{m+1}$ by removing a complete digraph $\overleftrightarrow{K}_2$ of order 2 (a zero matrix of order 2 in lower right).

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Bounds for max $\rho(D)$, II

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- For $l \geq 2$, there exists a constant C_l such that if $m \geq C_l$, a digraph $D^* \in \mathcal{D}(m^2 + l)$ satisfying

$$\rho(D^*) = \max\{\rho(D) : D \in \mathcal{D}(m^2 + l)\}$$

is obtained from $\overleftrightarrow{K_m}$ by including a new vertex u and edges in both directions joining u and $\lfloor l/2 \rfloor$ vertices of $\overleftrightarrow{K_m}$, and, if l is odd, a edge in either direction joining u and an additional vertex of $\overleftrightarrow{K_m}$.

min $\rho(D)$ with e edges

- If $e \leq \binom{n}{2}$, then $\widetilde{\rho}(n,e) = 0$: there is a digraph $D \in \mathcal{D}_n(e)$ such that every edge is of the form (i,j) with $i > j$ (the adjacency matrix has 0s on and above the main diagonal).
- If
$$\binom{n}{2} < e \leq \binom{n+1}{2},$$
then $\widetilde{\rho}(n,e) = 1$: there is a digraph whose adjacency matrix has 0s above the main diagonal and at least one 1 on the main diagonal).

So assume that $e > \binom{n+1}{2}$.

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e large ($\geq 0.75n^2$)

B+Solheid (1986)
Let $n \geq 2$ and $0 \leq \tau \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then

$$\widetilde{\rho}(n, n^2 - \tau) = \frac{n + \sqrt{n^2 - 4\tau}}{2}.$$

If $D \in \mathcal{D}_n(n^2 - \tau)$, then $\rho(D) = \widetilde{\rho}(n, n^2 - \tau)$ iff there are p and q with $p + q = n$, such that D is obtained by taking the complete digraph $\overleftrightarrow{K_n}$, partitioning its vertices into sets U and W of cardinalities p and q, respectively, and then removing any τ edges from the vertices in U to the vertices in W.

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B+Solheid (1986)
Let $n \geq 2$ and $0 \leq \tau \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then

$$\widetilde{\rho}(n, n^2 - \tau) = \frac{n + \sqrt{n^2 - 4\tau}}{2}.$$

If $D \in \mathcal{D}_n(n^2 - \tau)$, then $\rho(D) = \widetilde{\rho}(n, n^2 - \tau)$ iff there are p and q with $p + q = n$, such that D is obtained by taking the complete digraph $\overset{\longleftrightarrow}{K_n}$, partitioning its vertices into sets U and W of cardinalities p and q, respectively, and then removing any τ edges from the vertices in U to the vertices in W.

Example

If $n = 7$ and $\tau = 6$, an example with equality is the digraph with adjacency matrix

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

The other e's

In the remaining cases, $\widetilde{\rho}(n, n^2 - \tau)$ can be sandwiched between two consecutive integers: Let $1 \leq k \leq n$, and let $n = qk + l$ where q is a positive integer and $0 \leq l < k$. Define

$$\tau_{n,k} = \frac{q(q-1)}{2}k^2 + qkl.$$

- Let $0 \leq \tau < \binom{n}{2}$. Let $1 \leq k \leq n-1$ be such that

$$\tau_{n,k+1} \leq \tau < \tau_{n,k}.$$

Then

$$k < \widetilde{\rho}(n, n^2 - \tau) \leq k + 1.$$

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Adding edges to Δ_n

Δ_n is the **transitive tournament** on n vertices: there is an edge from i to j iff $n \geq i > j \geq 1$. For example, Δ_5 has adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Δ_n has all eigenvalues equal to 0 and so spectral radius 0.

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If $d = 1$, the maximum spectral radius occurs only by putting the new edge from 1 to n:

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General Theorem for Max Spectral Radius

Theorem: Let d be a positive integer. Then for n sufficiently large, a digraph with maximal spectral radius obtained by adding d new edges to Δ_n has the property that the new edges (1s) have an **upper staircase pattern**. For example, with $n = 7$ and $d = 8$, one possibility is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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Minimum Spectral Radius

Theorem:

- For $1 \leq d \leq n(n-1)/2$, the minimum spectral radius is attained by a matrix with a **staircase pattern**.

An example of such a matrix with a staircase pattern, with $n = 7$ and $d = 8$ is:

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Bounds on Spectral Radius

There is an impressive series of three papers by **L. Kolotilina** (2005-06-06) that extends and generalizes many classical bounds for the spectral radius of a nonnegative matrix. When specialized to digraphs they give very interesting conclusions. For instance:

Theorem Let D be a digraph of order n with a positive outdegree vector $R = (r_1, r_2, \ldots, r_n)$. Then for each α with $0 \leq \alpha \leq 1$,

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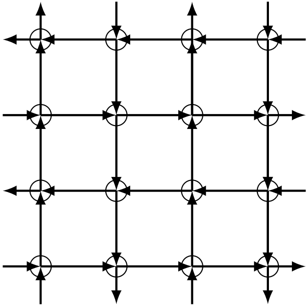
Manhattan Street Digraph

The **Manhattan street digraph** (view on a torus) $M_2(4,4)$ is:



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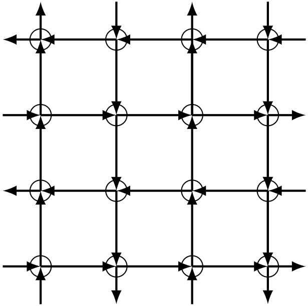

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Spectrum of $M_2(n_1, n_2)$

Theorem (Comellas, Dalfó, Fiol, 2008): The eigenvalues of $M_2(n_1, n_2)$ are

$$0, \pm\sqrt{2\cos\left(\frac{4\pi k}{n_1}\right) + 2\cos\left(\frac{4\pi l}{n_2}\right)} \quad \left(0 \le k \le \frac{n_1}{2} - 1, 0 \le l \le \frac{n_2}{2} - 1\right)$$

In addition, the geometric multiplicity of each nonzero eigenvalue equals its algebraic multiplicity, while the geometric multiplicity of the eigenvalue 0 is at least $(n_1 n_2)/2$, and equals $(n_1 n_2)/2$ if $n_i \not\equiv 0 \mod 4$ for $i = 1$ and 2.

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Wrapped Butterfly Digraph

Wrapped butterfly digraphs have been studied for their application in network theory.

Let d and n be positive integers. The **wrapped butterfly digraph** $B_d(n)$ has **vertices**

$$\{(l; x) = (l; x_0, x_1, \ldots, x_{n-1}) : 0 \le l \le n-1, 0 \le x_i \le d-1\}.$$

(The l in a vertex is called its **level**.)

Edges are:

$$(l; x_0, x_1, \ldots, x_{n-1}) \to (l+1; x_0, \ldots x_{l-1}, \alpha, x_{l+1}, \ldots, x_{n-1})$$

for every integer α with $0 \le \alpha \le d-1$. Here addition in the first component is modulo n and in the other components modulo d (thus the use of the word wrapped).

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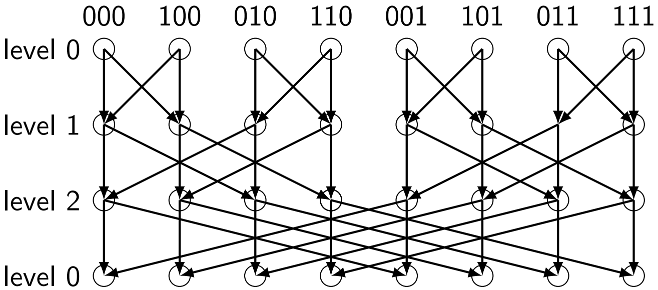

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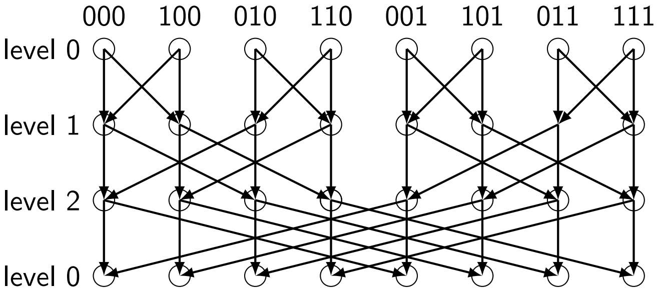

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Spectrum of $B_d(n)$

Theorem (Comellas, Fiol, Gimbert, Mitjana, 2008): The spectrum of the wrapped butterfly digraph $B_d(n)$ is

$$\mathbf{0}\ [n(d^n-1)], \mathbf{d}\ [1], \mathbf{d}\omega^1\ [1], \mathbf{d}\omega^2\ [1], \ldots, \mathbf{d}\omega^{\mathbf{n-1}}\ [1]$$

where $\omega = e^{2\pi i/n}$ and the quantities in the brackets are the algebraic multiplicities.

Tournaments

A **tournament** T is a digraph in which between each pair of distinct vertices there is exactly one edge (no loops).

The outdegree sequence of a tournament is usually called its **score sequence**. The adjacency matrix of a tournament is a **tournament matrix**.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$ is a tournament matrix.

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Tournament Spectra

Let T denote a tournament with score sequence

$$R = (r_1, r_2, \ldots, r_n) \quad \text{where } r_1 \leq r_2 \leq \cdots \leq r_n.$$

Then

$$A + A^t = J_n - I_n \quad (J_n \text{ the all 1s matrix}).$$

This equation leads to special spectral properties of tournaments.

For instance, the rank of a tournament matrix A of order n is at least $n-1$ and so if 0 is an eigenvalue of A, then it is a simple eigenvalue.

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Theorem Brauer and Gentry (Bull. AMS: 1968, LAA: 1972) The real part of each eigenvalue of a tournament T of order n is at least $-1/2$. Moreover,

$$\min\{(r_1 r_2 r_3)^{1/3}, (r_1 r_3)^{1/2}\} \leq \rho(T) \leq \frac{n-1}{2}.$$

Equality occurs on the right if and only if T is a regular tournament (and so n must be odd).

Also, for each eigenvalue λ of T,

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Maximum Spectral Radius for n even

Conjecture: RAB and Li (Disc. Math: 1983) For n even, the maximum spectral radius $\overline{\rho}_n$ of a tournament of order n equals the spectral radius of the **nearly regular tournament** with adjacency matrix

$$\left[\begin{array}{c|c} L_{n/2} & L_{n/2}^t + I_{n/2} \\ \hline L_{n/2}^t & L_{n/2} \end{array}\right], \text{ where } L_{n/2} = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

is the adjacency matrix of the transitive tournament of order $n/2$. These tournaments have been called Brualdi-Li tournaments.

Progress on the Conjecture for $\overline{\rho}_n$, n even

Theorem: Kirkland (LAMA: 1991, LAA: 1997, LAA:2003): If T is a nearly regular tournament of order $n = 2m$, then

$$\rho(T) \geq \frac{m-1}{2} - \sqrt{\frac{m^2-1}{4}}.$$

For every regular tournament of order m with adjacency matrix S, the nearly regular tournament T of order n with adjacency matrix

$$\begin{bmatrix} S & S^t \\ S^t + I_m & S \end{bmatrix}$$

has this spectral radius

Progress on the Conjecture for $\overline{\rho}_n$, n even

(Kirkland continued)

If n is even, then

$$\overline{\rho}_n = \frac{n-1}{2} - \frac{\gamma_n}{n} + O\left(\frac{1}{n^2}\right), \text{ where}$$

$$0.377453\ldots \approx \frac{2 \cdot 3^{2/3} - 3^{4/3} + 13}{34} \leq \gamma_n \leq \frac{e^2 - 1}{2(e^2+1)} \approx 0.380797\ldots.$$

Moreover, for n sufficiently large a tournament of order n with maximum spectral radius must be nearly regular.

Minimum Spectral Radius of Tournaments

Conjecture: RAB and Li (Disc. Math: 1983) Let $\widetilde{\rho}_n$ denote the minimum spectral radius of a **strongly connected** tournament of order n. Then $\widetilde{\rho}_n$ equals the spectral radius of the tournament $\widetilde{T}_n$ with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

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Theorem: Kirkland (LAA: 1996) Let T be a strongly connected tournament of order n. Then

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with equality if and only if T is isomorphic to $\widetilde{T}_n$.

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Example

Consider the digraph D with adjacency matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

All square submatrices have a nonnegative determinant; thus A, respectively, D, is **totally nonnegative**.

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Total nonnegativity of a digraph depends on the order in which the vertices are listed.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is totally nonnegative, but interchanging rows 1 and 2 and interchanging columns 1 and 2 gives the matrix

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A Theorem

Theorem (McKay et al, 2004) If all the eigenvalues of a digraph D are positive, then the digraph has no cycles of length > 1.

Such a digraph has an adjacency matrix with all 1s on the main diagonal and all 0s above the main diagonal; thus **all eigenvalues equal 1**. This digraph need not be totally nonnegative: recall the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

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Theorem (McKay et al, 2004) If all the eigenvalues of a digraph D are positive, then the digraph has no cycles of length > 1.

Such a digraph has an adjacency matrix with all 1s on the main diagonal and all 0s above the main diagonal; thus **all eigenvalues equal 1**. This digraph need not be totally nonnegative: recall the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Corollary An **irreducible** $(0,1)$-matrix of order $n \geq 2$ with all eigenvalues nonnegative is singular (0 must be an eigenvalue).

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Theorem Generalized

Theorem Let D be a digraph of order n with r positive eigenvalues and $n-r$ zero eigenvalues. Assume that D has exactly r loops. Then D has no cycles of length greater than 1.

Proof Outline: A the adjacency matrix of D with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Then

$$1 = \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \cdots + \lambda_r}{r} \geq \left(\prod_{i=1}^{r} \lambda_i\right)^{1/r} \geq 1.$$

This implies that $\lambda_1 = \cdots = \lambda_r = 1$. Now use the P-F theory to conclude that A has r irreducible components equal to $I_1 = [1]$ and $n-r$ irreducible components equal to $O_1 = [0]$.

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Remark

It is not hard to show that if A has $n-1$ positive and one zero eigenvalue, then the trace of A equals $n-1$ or n. The preceding theorem takes care of the case of trace equal to $n-1$. If trace of A equals n, A need not be triangularizable. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (\text{trace} = 2, \text{ eigenvalues } 0, 2), \text{ and}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} (\text{trace} = 3, \text{ and eigenvalues } 0, 0.3820, 2.6180).$$

Totally Nonnegative Digraphs

Theorem (RAB and Kirkland, 2009): An m by n (0,1)-matrix is totally nonnegative iff it has no submatrix equal to one of

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

More on totally nonnegative digraphs ((0,1)-matrices) in a forthcoming paper by RAB and Kirkland. E.g. **An irreducible totally nonnegative $(0,1)$-matrix of order n has 0 as an eigenvalue of multiplicity at least $\lfloor n/2 \rfloor$.**

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Definition and Example

Digraphs D_1 and D_2 are **cospectral** provided they are not isomorphic but have the same spectrum (may be complex).

Example (Krishnamurthy and Parthsarathy, 1974, 1975): Let D_i be the digraph obtained from a directed cycle $(v_1, v_2, \ldots, v_{2k}, v_1)$ of length $2k$ by adding two new vertices, u and w, and inserting edges $(v_1, u), (u, v_1), (w, v_i), (v_i, w)$. Then for $2 \leq i \leq k+1$, the resulting strongly connected digraphs of order $n = 2k + 2$ are cospectral, indeed have characteristic polynomial equal to

$$\lambda^{2k+2} - 2\lambda^{2k} + \lambda^{2k-2} - \lambda^2.$$

Note: It is easy to find non-strongly connected examples.

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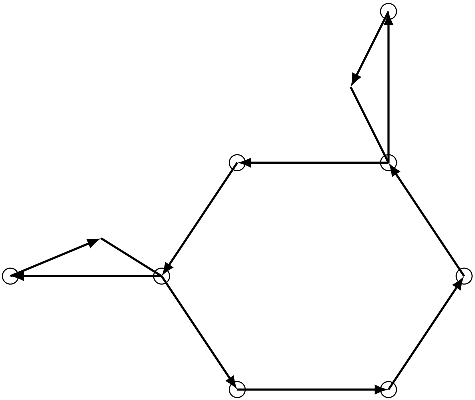
Previous Example with $k = 6$



Three cospectral, strongly connected digraphs of this type.

Energy

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Nikiforov (2007) defined the **energy** of a matrix A to be the sum of the singular values of A (the positive square roots of the eigenvalues of the p-s-d symmetric matrix AA^T): $E(A) = \sigma_1 + \sigma_2 + \cdots$, and gave a general upper bound for $E(A)$ which for digraphs with q edges becomes:

$$E(D) \leq \frac{q}{n} + \sqrt{(n-1)\left(q - \frac{q^2}{n^2}\right)}$$

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Peña and Rada (2008) gave another definition of energy—called here **low energy**—using eigenvalues not singular values:

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Coulson's integral formula for energy of graphs is extended to low energy of digraphs and the McClelland identity is extended:

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where c_2 is the number of closed walks in D of length 2. Equality holds if and only if D is the direct sum of $n/2$ copies of $\overleftrightarrow{K_2}$.

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Laplacian matrix of a digraph

Chung (2005) introduced **Laplacians** of (strongly connected) digraphs D. Outdegrees: $r_1, r_2, \ldots, r_n$; Indegrees: $s_1, s_2, \ldots, s_n$. Let $P = [p_{ij}]$ be the matrix of order n defined by

$$p_{ij} = \begin{cases} \frac{1}{r_i} & \text{if } (v_i, v_j) \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

P is the (irreducible) transition matrix of a random walk on D (all row sums are 1 and $\rho(P) = 1$), and P has a unique, normalized left eigenvector ϕ for 1: $\phi P = \phi$, $\sum_{i=1}^{n} \phi_i = 1$.

$$\mathcal{L}(D) = I_n - \frac{\Phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^T \Phi^{1/2}}{2},$$

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Laplacian spectrum of a digraph

If D is a symmetric digraph (a graph), then $\phi = \frac{1}{d}(r_1, r_2, \ldots, r_n)$ where $d = \sum_{i=1}^{n} r_i$. $\mathcal{L}(D)$ is the symmetric matrix $I_n - XAX$ where A is the adjacency matrix of D and

$$X = \mathrm{diag}\left(\frac{1}{\sqrt{r_1}}, \frac{1}{\sqrt{r_2}}, \ldots, \frac{1}{\sqrt{r_n}}\right).$$

Thus, the Laplacian of a symmetric digraph is the so-called *normalized Laplacian*. $\mathcal{L}(D)$ is a singular, positive semidefinite symmetric matrix with eigenvalues $\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, called the **Laplacian eigenvalues** or **Laplacian spectrum** of D.

Laplacian spectrum and Diameter

Theorem (Chung 2006): The diameter of D is at most

$$\left\lfloor \frac{2\min\left\{\log\left(\frac{1}{\phi_i}\right) : 1 \leq i \leq n\right\}}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1,$$

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(This generalizes upper bounds on diameter of a graph using eigenvalues.)

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