
A New Hybrid Method for the Quadratic Affine Inverse Eigenvalue Problem

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Free Vibrations

Those matrices have very important features:

M, C, K are **symmetric**

M is **positive definite**

C, K are **semi-definite**

They are **structured**

The motion of the system is identified by

Natural Frequencies = square roots of eigenvalues of $P(\lambda)$

Mode Shapes = Eigenvectors of $P(\lambda)$

$$\begin{cases} M\ddot{q} + C\dot{q} + Kq = 0 \\ q(0) = q_0 \\ \dot{q}(0) = q_1 \end{cases}$$

$$q \in \mathbb{R}^n \text{ and } M, C, D \in \mathbb{R}^{n \times n}$$

$$q(t) = xe^{\lambda t}$$

$$P(\lambda)x = 0$$

$$P(\lambda) = \lambda^2 M + \lambda C + K$$

Affine Inverse Eigenvalue Problem

$$P(\lambda) = \lambda^2 M + \lambda C + K$$

$$C = C(\alpha) = C_0 + \sum_{i=1}^n \alpha_i C_i, \quad K = K(\beta) = K_0 + \sum_{i=1}^n \beta_i K_i,$$

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}, \quad \beta = (\beta_1, \dots, \beta_n)$$

$$(M, C(\alpha), K(\beta))$$

C_i, K_i, M , are real, $n \times n$ symmetric matrices

Eigenvalues: $\{\lambda_1(\alpha, \beta), \dots, \lambda_{2n}(\alpha, \beta)\}$

Problem 1. Given a set with distinct entries $\{\lambda_1^*, \dots, \lambda_{2n}^*\}$, closed under complex conjugation, find $(\alpha, \beta) \in \mathbb{R}^{2n}$, such that $\{\lambda_1^*, \dots, \lambda_{2n}^*\}$ are eigenvalues of the pencil $(M, C(\alpha), K(\beta))$.

We will denote $\{\lambda_1^*, \dots, \lambda_{2n}^*\}$ the target set of eigenvalues.

Reformulation of the Problem

$$(C, K) \in \mathbf{R}^{2n^2}, \langle (C_1, K_1), (C_2, K_2) \rangle = \text{trace}(C_1^T C_2 + K_1^T K_2)$$

$$\mathcal{A} = \{(C, K) \in \mathbf{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^n \alpha_i C_i, K = K(\beta) = K_0 + \sum_{i=1}^n \beta_i K_i\}.$$

and

$$\mathcal{L} = \{(C, K) \in \mathbf{R}^{2n^2} \mid MX(\Lambda^*)^2 + CX\Lambda^* + KX = 0, \text{ for some matrix } X \text{ s.t. } \|x_i\| = 1\},$$

where $\Lambda^* = \text{diag}(\lambda_1^*, \dots, \lambda_{2n}^*)$

Problem 1 can be reformulated:

Problem 2. Find $(\alpha, \beta) \in \mathbf{R}^{2n}$ such that $(M, C(\alpha), K(\beta)) \in \mathcal{L} \cap \mathcal{A}$.

Alternating Projections

$$\mathcal{A} = \{(C, K) \in \mathbb{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^n \alpha_i C_i, K = K(\alpha) = K_0 + \sum_{i=1}^n \beta_i K_i\}$$

and

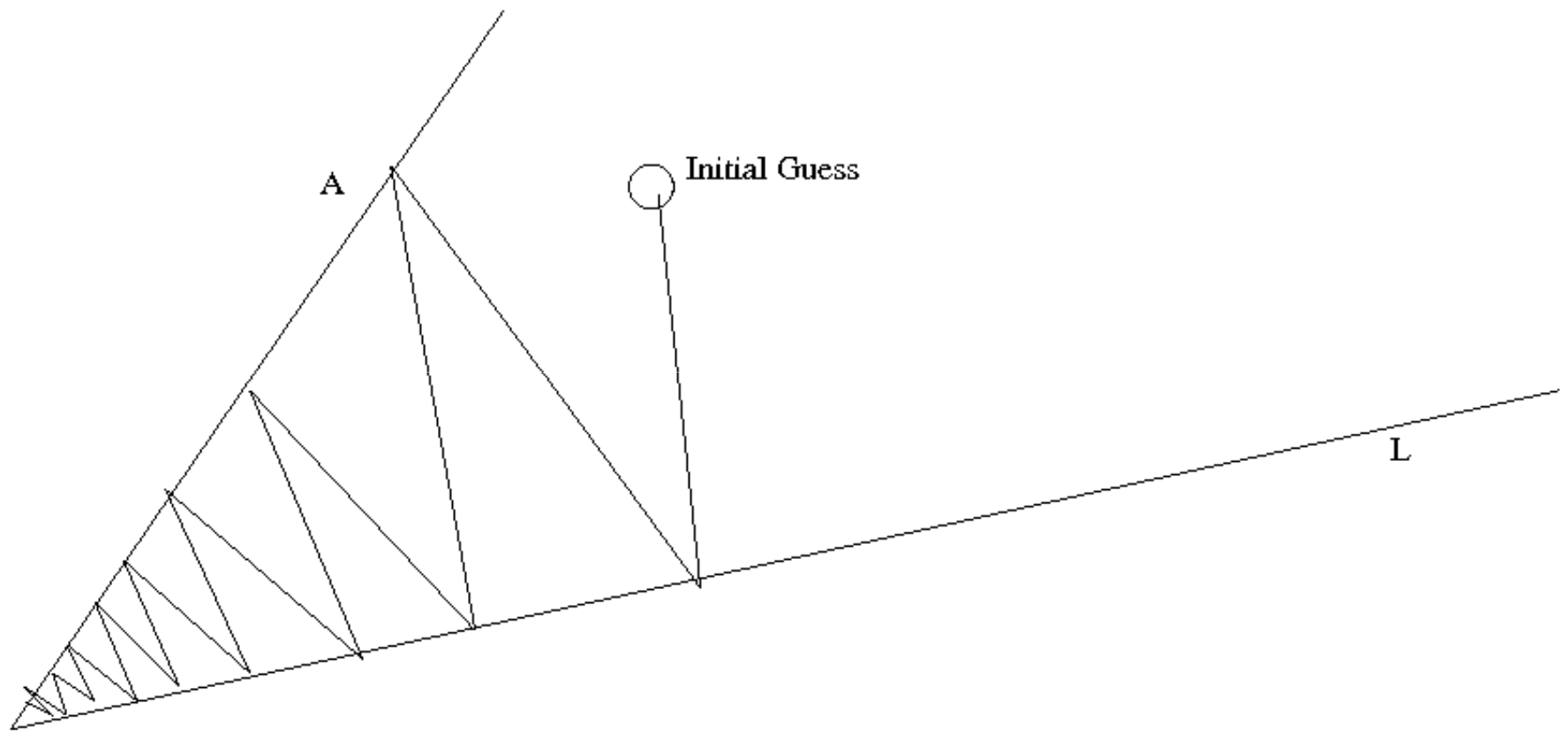
$$\mathcal{L} = \{(C, K) \in \mathbb{R}^{2n^2} \mid MX(\Lambda^*)^2 + CX\Lambda + KX = 0, \text{ for some matrix } X \text{ s.t. } \|x_i\| = 1\}$$

$P_{\mathcal{A}}(C, K)$ – projection onto \mathcal{A}

$P_{\mathcal{L}} : \mathbb{R}^{2n^2} \rightarrow \mathcal{L}$, $P_{\mathcal{L}}(C, K) = (X, Y)$ – projection onto \mathcal{L}

$$P_{\mathcal{L}}(C, K) \in \arg \min_{(X, Y) \in \mathcal{L}} \|X - C\| + \|Y - K\|$$

$$(P_{\mathcal{L}}P_{\mathcal{A}})^i(C_0, K_0) \rightarrow (C, K) \in \mathcal{L} \cap \mathcal{A}, i \rightarrow \infty$$



Convergence

Theorem A. *Let C_1, C_2 be closed convex sets in a finitely dimensional Hilbert space H , $C_1 \cap C_2 \neq \emptyset$ and let P_{C_1} and P_{C_2} denote projection operators onto C_1 and C_2 correspondingly. Then*

$$\lim_{n \rightarrow \infty} (P_{C_1} P_{C_2})^n = \lim_{n \rightarrow \infty} (P_{C_2} P_{C_1})^n = P_{C_1 \cap C_2}$$

Note, both of the sets are closed, however the set \mathcal{L} is nonconvex. Thus, alternating projections might not converge. However, alternating projection never increases the distance between successive iterates.

Theorem B. *Let C_1, C_2 be closed sets in a finitely dimensional Hilbert space*

H , $C_1 \cap C_2 \neq \emptyset$ and let $y \in C_2$. If

$$x_1 = P_{C_1}(y), \quad y_1 = P_{C_2}(x_1), \quad x_2 = P_{C_1}(y_1),$$

then

$$\|x_2 - y_1\| \leq \|x_1 - y_1\| \leq \|x_1 - y\|.$$

Corollary. *For any given $x_0 \in H$, $\{(P_{C_1} P_{C_2})^n(x_0)\}_{n=0}^{\infty}$ is a nondecreasing sequence.*

Global Convergence of the Algorithm

Theorem C. *Let C_1 and C_2 be closed sets, one of which is bounded.
 $\{(x_i, y_i)\}$ - a sequence generated by alternating projections, then
 \exists accumulation point (\bar{x}, \bar{y})*

- (\bar{x}, \bar{y}) lies in $C_1 \times C_2$
- $\|\bar{x} - \bar{y}\|_F = \lim_{j \rightarrow \infty} \|x_j - y_j\|_F$
- $\|\bar{x} - \bar{y}\|_F = \text{dist}(\bar{x}, C_1) = \text{dist}(\bar{y}, C_2)$

$P_{\mathcal{L}}$ - Matrix Nearness Problem

$$\mathcal{L} = \{(C, K) \in \mathbf{R}^{2n^2} \mid MX(\Lambda^*)^2 + CX\Lambda + KX = 0, \text{ for some matrix } X \text{ s.t. } \|x_i\| = 1\}$$

$$P_{\mathcal{L}}(C, K) \in \arg \min_{(X, Y) \in \mathcal{L}} \|X - C\| + \|Y - K\|$$

Let $(C, K) \in \mathbf{R}^{2n^2}$ and $(X_{\sigma}, \Lambda_{\sigma})$ be the eigenpair of the quadratic pencil (M, C, K) . Define $V = (V_1, V_2)$ as

$$V = \begin{pmatrix} X_{\sigma} \\ X_{\sigma}\Lambda^* \end{pmatrix}^{-1}$$

Then

$$P_{\mathcal{L}}(C, K) = (\hat{C}, \hat{K})$$

$$\hat{C} \approx -MX_{\sigma}(\Lambda^*)^2V_2$$

$$\hat{K} \approx -MX_{\sigma}(\Lambda^*)^2V_1$$

σ is the permutation which minimizes

$$\sum_{j=1}^{2n} |\lambda_{\sigma_j}(\alpha, \beta) - \lambda_j^*|$$

among all possible permutations of the list of eigenvalues $\lambda_i(\alpha, \beta)$.

$$\sigma \in \arg \min_{\sigma_1, \dots, \sigma_{2n}} \sum_{j=1}^{2n} |\lambda_{\sigma_j}(\alpha, \beta) - \lambda_j^*|$$

$$\Lambda_\sigma = \text{diag}(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_{2n}})$$

$$X_\sigma = (x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_{2n}})$$

Nonuniqueness is not important!

Projection onto \mathcal{A}

$$\mathcal{A} = \{(C, K) \in \mathbf{R}^{2n^2} \mid C = C(\alpha) = C_0 + \sum_{i=1}^n \alpha_i C_i, K = K(\alpha) = K_0 + \sum_{i=1}^n \beta_i K_i\}$$

Let $P_{\mathcal{A}}(C, K) = (C(\hat{\alpha}), K(\hat{\beta}))$, then coefficients $(\hat{\alpha}, \hat{\beta})$ could be found as solutions of the following linear systems

$$A_1 \hat{\alpha} = b_1, \quad A_2 \hat{\beta} = b_2,$$

where

$$(A_1)_{ij} = \text{trace}(C_i^T C_j),$$

$$(A_2)_{ij} = \text{trace}(K_i^T K_j),$$

$$(b_1)_i = \text{trace}((C - C_0)^T C_i),$$

$$(b_2)_i = \text{trace}((K - K_0)^T K_i).$$

Alternative Projections Algorithm

Algorithm 1 Alternative Projections Method

INPUT: M, C_i 's, K_i 's, λ^* , (α^0, β^0) , ϵ

OUTPUT: $(\hat{\alpha}, \hat{\beta})$

- 1: **for** $i = 0, 1, \dots$ **do**
 - 2: Form $(C(\alpha_i), K(\beta_i))$
 - 3: Compute eigenvalues-eigenvectors of $(M, C(\alpha_i), K(\beta_i))$ and compute σ
 - 4: Form matrix X_σ
 - 5: Compute (\hat{C}, \hat{K}) , projection of $(C(\alpha_i), K(\beta_i))$ onto \mathcal{L}
 - 6: Compute $P_{\mathcal{A}}(C, K)$, $(\alpha_{i+1}, \beta_{i+1})$
 - 7: Stop if $\|(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)\| < \epsilon$
 - 8: **end for**
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Inverse Iteration Step

Eigenvectors can be updated by one step of inverse iteration.

$$\begin{pmatrix} 0 & I \\ K & C \end{pmatrix} \begin{pmatrix} u_j \\ z_j \end{pmatrix} = \lambda_j^* \begin{pmatrix} I & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} x_i \\ \lambda_j^* x_i \end{pmatrix}, \quad i = 1, \dots, 2n.$$

We then define

$$x_{i+1} = \frac{u_j}{\|u_j\|}$$

which determines the new matrix X_{i+1} . Vector u_j can be obtained as a solution of an $n \times n$ linear system

$$(\lambda_j^{*2} M + \lambda_j^* C + K)u_j = (C + 2\lambda_j^* M)x_i, \quad i = 1, \dots, 2n$$

Newton's Method

Theorem *Let (α^*, β^*) be a solution to Problem 1 then there exist a neighborhood of (α^*, β^*) which contains no singular points. These are the points where the pencil has multiple eigenvalues.*

Corollary *There is a neighborhood of (α^*, β^*) where $\lambda_i(\alpha, \beta)$ are distinct and are differentiable functions.*

$$f(\alpha, \beta) = \begin{pmatrix} \lambda_{\sigma(1)}(\alpha, \beta) - \lambda_1^* \\ \vdots \\ \lambda_{\sigma(2n)}(\alpha, \beta) - \lambda_{2n}^* \end{pmatrix} = 0.$$

Newton's Step

$$J(\alpha^i, \beta^i) \begin{pmatrix} \alpha^{i+1} - \alpha^i \\ \beta^{i+1} - \beta^i \end{pmatrix} = -f(\alpha^i, \beta^i)$$

$$J_{ik} = \begin{cases} \frac{\partial \lambda_i}{\partial \alpha_k} = -\frac{\lambda_i x_i^T C_k x_i}{x_i^T (2\lambda_i M + C) x_i}, & k = 1, \dots, n \\ \frac{\partial \lambda_i}{\partial \beta_{k-n}} = -\frac{x_i^T K_{k-n} x_i}{x_i^T (2\lambda_i M + C) x_i}, & k = n + 1, \dots, 2n \end{cases}$$

Newton's Algorithm

Algorithm 1 Newton's Method

INPUT: λ^* , (α_0, β_0) , tolerance - ϵ

OUTPUT: (α^*, β^*)

- 1: **for** $i = 0, 1, \dots$ **do**
 - 2: Find eigenvalues and eigenvectors of $(M, C(\alpha_i), K(\beta_i))$.
 - 3: Solve the minimization combinatorics problem, compute σ
 (can be done by Hungarian Method in $O(n^3)$)
 - 4: Compute $f(\alpha_i, \beta_i)$
 - 5: Form $J(\alpha^i, \beta^i)$ and find $(\alpha^{i+1}, \beta^{i+1})$
 - 6: Stop if $\|(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)\| < \epsilon$
 - 7: **end for**
-

Convergence

Conditions for quadratic convergence are satisfied:

- f is differentiable and J is Lipschitz continuous in a neighborhood of (α^*, β^*)
- $J(\alpha^*, \beta^*)$ is nonsingular

Algorithm

Algorithm 1 Alternative Projections - like Method

INPUT: λ^* , (α^0, β^0) , ϵ

OUTPUT: $(\hat{\alpha}, \hat{\beta})$

- 1: Form $(C(\alpha_0), K(\beta_0))$
- 2: Compute eigenvalues-eigenvectors of $(M, C(\alpha_0), K(\beta_0))$ and compute σ
- 3: Form matrix $X_\sigma^{(0)} = X_\sigma(\alpha_0, \beta_0)$
- 4: **for** $i = 0, 1, \dots$ **do**
- 5: Compute (\hat{C}, \hat{K}) , quasi-projection of $(C(\alpha_i), K(\beta_i))$ onto \mathcal{L}
- 6: Compute $(\alpha_{i+1}, \beta_{i+1})$
- 7: Stop if $\|(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)\| < \epsilon$
- 8: Form $(C, K) = (C(\alpha_{i+1}), K(\beta_{i+1}))$
- 9: Solve $2n$ linear systems

$$((\lambda_j^*)^2 M + \lambda_j^* C + K)u_j = (C + 2\lambda_j^* M)x_{\sigma(j)}^i, \quad j = 1, \dots, 2n$$

and compute

$$x_{\sigma(j)}^{i+1} = \frac{u^j}{\|u^j\|}$$

10: **end for**

Hybrid Method

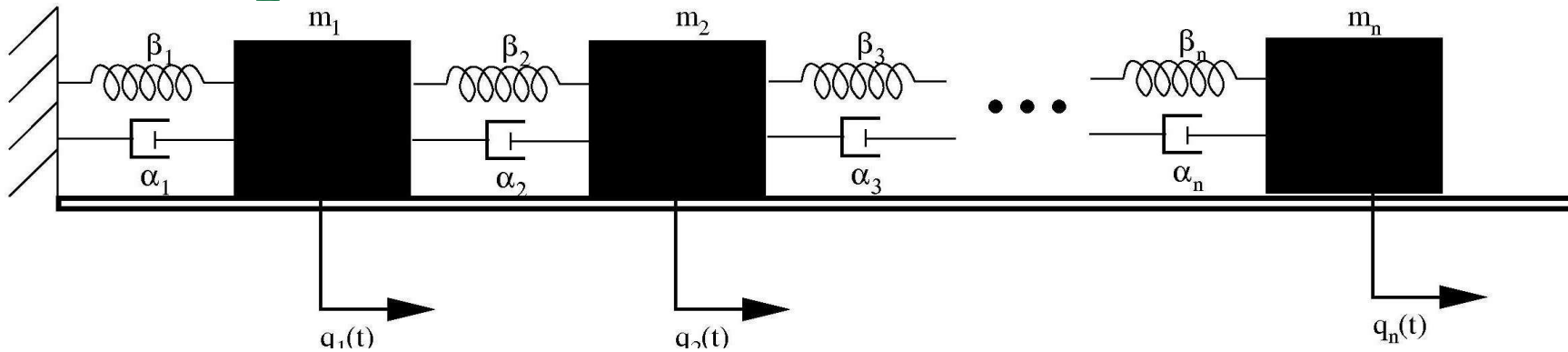
Algorithm 1 Hybrid Method

INPUT: λ^* , (α^0, β^0) , ϵ_1 , ϵ_2

OUTPUT: (α^*, β^*)

- 1: **while** $\|(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)\| < \epsilon_2$ **do**
 - 2: $(\alpha^{i+1}, \beta^{i+1}) = c((P_{\mathcal{A}}P_{\mathcal{L}})(C(\alpha^i), K(\beta^i)))$
 - 3: **end while**
 - 4: **while** $\|(\alpha^{i+1}, \beta^{i+1}) - (\alpha^i, \beta^i)\| < \epsilon_1$ **do**
 - 5: Form $J(\alpha^i, \beta^i)$ and compute $(\alpha^{i+1}, \beta^{i+1})$
 - 6: **end while**
-

Example



$$K(\beta) = \begin{pmatrix} \beta_1 + \beta_2 & -\beta_1 & & & \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 & & \\ & -\beta_3 & \beta_3 + \beta_4 & -\beta_4 & \\ & & \ddots & \ddots & \ddots \\ & & & -\beta_n & \beta_n \end{pmatrix}$$

$$K_0 = 0, \quad K_1 = e_1 e_1^T$$

$$K_i = (e_{i-1} - e_i)(e_{i-1} - e_i)^T, \quad i = 2, 3, \dots, n$$

$$K = K(\beta) = K_0 + \sum_{i=1}^n \beta_i K_i$$

Numerical Experiment

$$\alpha_0 = (1, 1, 1), \beta_0 = (1, 1, 1)$$

$$\lambda^* = \{-0.0271 \pm i1.0108, -0.0177 \pm i0.6724, -0.0023 \pm i0.2658\}$$

Solution of Modified Alternating Projection Method:

$$\alpha_{AP} = (0.0332, 0.0134, 0.0169), \beta_{AP} = (0.7188, 0.2193, 0.1915)$$

$$\lambda(\alpha_{AP}, \beta_{AP}) = \{-0.0271 \pm i1.0110, -0.0176 \pm i0.6724, -0.0021 \pm i0.2554\}$$

Locally Unique Solution of Newton's Method:

$$\alpha_N = (0.0139, 0.0203, 0.0199), \beta_N = (0.6038, 0.2722, 0.1988)$$

$$\|\lambda^* - \lambda(\alpha_N, \beta_N)\| = 1.29 \times 10^{-9}$$