A New Hybrid Method for Finding Eigenpairs of Symmetric Quadratic Eigenvalue Problem in an Interval

Karabi Datta, Yoopyo Hong, Mohan Thapa

Northern Illinois University
Department of Mathematical Sciences
DeKalb, IL 60115

Overview

Introduction(QEP)

The behavior of the physical phenomenon in different applications can be described by a second-order differential equation as shown,

$$\begin{aligned} f(t) &= M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \\ q(0) &= g \\ \dot{q}(0) &= h \end{aligned} \tag{1}$$

where $f(t)$ is a time dependent external force vector
g, h are initial condition vectors at time $t = 0$
M = *mass matrix*,
C = *damping matrix* and
K = *stiffness matrix*.

Introduction(QEP)

$$0 = M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \tag{2}$$

The solution of the above homogeneous second-order differential equation can be expressed,

$$q(t) = e^{\lambda t}u,$$

which is the eigensolution of the corresponding quadratic eigenvalue problem(QEP),

$$(\lambda^2 M + \lambda C + K)u = 0 \tag{3}$$

where, λ = *eigenvalue*, and
u = *eigenvector* corresponding to the eigenvalue λ.

Some Applications

1. Vibration Analysis of Structural Systems
2. Constrained least squares problem
3. Eigenvalue assignment problem for quadratic matrix pencil

Vibration Analysis of Structural Systems

In practice response $q(t)$ of a system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t)$$

excited by a time harmonic force $f(t) = f_0 e^{iw_0 t}$ with some frequency w_0. As iw_0 approaches to eigenvalue of the system, response $q(t)$ blows. This case we call it *resonance* condition.

Constrained least squares problem

$$\min_{x^T x = \alpha^2} \{x^T A x - 2b^T x\}$$

where $A = A^T \in \mathbb{R}^{n \times n}$, $b = (Ax - \lambda x) \in \mathbb{R}^n$.
The solution is $x = (A - \lambda I)^{-1} b$, where λ is the smallest eigenvalue of

$$\left(\lambda^2 I + 2\lambda A + (A^2 - \alpha^{-2} b b^T)\right) y = 0,$$

where $y = (A - \lambda I)^{-2} b$.

Partial eigenvalue assignment problem for quadratic matrix pencil

This is the problem of reassigning a part of the spectrum of quadratic system by feedback control, leaving the rest of the spectrum invariant.

A Brief Review of the Existing Methods

1. Linearization
2. Jacobi-Davidson Method for *QEP*
3. Second-Order Arnoldi Method

Linearization

The Quadratic eigenvalue problem of the form $(\lambda^2 M + \lambda C + K)u = 0$ is usually solved in two stages.

Stage I. Transform QEP into equivalent *Standard Eigenvalue problem*

$$Ay = \lambda y$$

where

$$A = \begin{pmatrix} -M^{-1}C & -M^{-1}K \\ I & O \end{pmatrix} \text{ and } y = \begin{pmatrix} \lambda u \\ u \end{pmatrix}$$

(M is a non singular $n \times n$ matrix).

Linearization

Stage II.

Compute the Eigenpairs of the *Standard Eigenvalue Problem*.

Then find the eigenpairs by using *QR* algorithm if *n* is small and dense, or apply Krylov-subspace-based methods to find few eigenvalues of the *QEP*.

Jacobi-Davidson Method for QEP[SLEIJPEN 8]

This method also consists of *two* major steps.

First step

The projection of the higher dimension QEP to a low dimension subspace, by choosing suitable V,

$$(\alpha^2 V^* MV + \alpha V^* CV + V^* KV)s = 0$$

where α is an eigenvalue and
$x \equiv Vs$) is eigenvector corresponding to the eignvalue α.
The pair (α, x) is called the **Ritz pair**.

If the norm of the residual $\| (Q(\alpha)x) \|$ is larger than tolerance then go to step II.

Jacobi-Davidson Method for QEP

Second Step

The expansion of the search subspace, span(V).

Compute orthogonal correction vector t for Ritz vector x_k, so that we have

$$(\alpha^2 M + \alpha C + K)(x_k + t) = 0.$$

As $t \perp x_k$ the operator $Q(\lambda)$ can be restricted to the subspace orthogonal to x_k and solve for t,

$$\left(I - \frac{p x_k^*}{x_k^* p}\right) Q(\alpha)(I - x_k x_k^*)t = -r_k$$

where, $r_k = Q(\alpha)x_k$ is *residual* and
$p = Q'(\alpha)x_k$, with $Q'(\alpha) = 2\alpha M + C$.

Second-Order Arnoldi Method(SOAR)[BAI 11]

Second-order Krylov subspace $G_k(A, B; u)$ induced by a pair of matrices $A,\ B \in \mathbb{R}^{n \times n}$ and a vector $u \in \mathbb{R}^n$ is defines as,

$$G_k(A, B; u) = span\{r_0, r_1, \ldots, r_{k-1}\},$$

where

$$\begin{aligned} r_0 &= u, \\ r_1 &= Ar_0, \\ r_j &= Ar_{j-1} + Br_{j-2},\ for\ \ j \geq 2, \end{aligned}$$

and $A = -M^{-1}C$, $B = -M^{-1}K$
We have,

$$(\alpha^2 M + \alpha C + K)z = 0$$

where (α, z) is a approximate eigenpair.

Second-Order Arnoldi Method

- In $m << n$ steps, SOAR procedure with $A = -M^{-1}C$, $B = -M^{-1}K$ and a starting vector u generates an $n \times m$ matrix Q_m which is an orthonormal basis of second-order krylov subspace $G_m(A, B; u)$.
- compute $M_m = Q_m^T M Q_m,\ \ C_m = Q_m^T C Q_m,\ \ K_m = Q_m^T K Q_m$.

  Now we have new reduced system of order m,

  $$(\alpha^2 M_m + \alpha C_m + K_m)g = 0$$

  where (α, g) is an eigenpair of reduced QEP.
  Obtain the *Ritz pairs* (α, z) where $z = \dfrac{Q_m g}{||Q_m g||}$
- Check the residual with the new *Ritz pairs* (α, z).

Drawback of the Existing Methods

Linearized method

- The generalized eigenvalue problem is twice the dimension of the original QEP.
- One of the drawback of the linearization process is the lost of the positive definiteness. Subsequently the essential spectral properties of QEP are not guaranteed to be preserved.

Drawback of the Existing Methods

Jacobi-Davidson Method

- Successes of the Jacobi-Davidson Method strongly depends how to choose the initial vectors.
- This method usually converges to largest eigenpair. Hence it is not suitable to apply this method to find targeted eigenpair in a given interval.

Second-order Arnoldi method

- Second-order Arnoldi method does not give good approximation of eigenpair closer to the lower part of the spectrum.
- It is very difficult to choose size m to find the invariant subspace.
- Again with this method it is not easy to find eigenvalues in a specific interval.

The New Hybrid Method

The Hybrid method has *two* parts.

First part

We choose $m << n$ sets of random eigenpairs
where $\alpha_i \in [a,\ b]$, and
the orthogonal vectors $v_i,\ \ i = 1,\ 2, ..., m$,
then run few-iterations of Modified Parametrized Newton's method for each pair.

Second part

Use the above eigenvectors and one of the eigenvalue as an initial eigenvectors and shift to run the Jacobi-Davidson method.

The Modified Parametrized Newton(MPN) Method to the Symmetric QEP

For $\quad f(u,\lambda) = \begin{pmatrix} Q(\lambda)u \\ u^T u - 1 \end{pmatrix} = \begin{pmatrix} (\lambda^2 M + \lambda C + K)u \\ u^T u - 1 \end{pmatrix}$

Define $\quad f_s(u,\lambda) = \begin{pmatrix} Q_s(\lambda)u \\ u^T u - 1 \end{pmatrix}$

$$= \begin{pmatrix} (\lambda((\alpha_1+\alpha_2)M + C) - \alpha_1\alpha_2 M + K)u \\ u^T u - 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda R_s + T)u \\ u^T u - 1 \end{pmatrix},$$

where $R_s = [(\alpha_1+\alpha_2)M + C]$ and $T = [-\alpha_1\alpha_2 M + K]$.

MPN Method

Let $Q_s(\lambda)u = (\lambda R_s + T)u$.
Jacobian matrix J_{f_s} of f which can be calculated as

$$J_{f_s}(u,\lambda) = \begin{pmatrix} \lambda R_s + T & R_s u \\ 2u^T & 0 \end{pmatrix}$$

notice $R_s = (\alpha_1 + \alpha_2)M + C = Q_s'(\lambda)$ is the derivative of the quadratic matrix pencil $Q_s(\lambda)$

MPN Method

MPN iteration is

$$\begin{pmatrix} x^{(i+1)} \\ \alpha^{(i+1)} \end{pmatrix} = \begin{pmatrix} x^{(i)} \\ \alpha^{(i)} \end{pmatrix} - \begin{bmatrix} Q_s(\alpha^{(i)}) & R_s^{(i)} x^{(i)} \\ 2{x^{(i)}}^T & 0 \end{bmatrix}^{-1} \begin{pmatrix} Q_s(\alpha^{(i)}) x^{(i)} \\ {x^{(i)}}^T x^{(i)} - 1 \end{pmatrix}$$

Now choose a parameter $\tilde{t} > 0$ and assume $\alpha^{(i)} \neq 0$ so that the method takes the form

$$\begin{bmatrix} Q_s(\alpha^{(i)}) & R_s^{(i)} x^{(i)} \\ 2{x^{(i)}}^T & 0 \end{bmatrix} \begin{pmatrix} x^{(i+1)} \\ \alpha^{(i+1)} \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{t} \end{bmatrix} \begin{bmatrix} 0 & R_s^{(i)} x^{(i)} \\ {x^{(i)}}^T & \frac{1}{\alpha^{(i)}} \end{bmatrix} \begin{pmatrix} x^{(i)} \\ \alpha^{(i)} \end{pmatrix}$$

MPN Method

Modified Parametrized Newton's iteration for $Q(\lambda)$ now takes the form

$$\begin{aligned} \alpha^{(i+1)} &= \alpha^{(i)} - \frac{(r^{(i)})^2}{\hat{\beta}^{(i)}} s \\ x^{(i+1)} &= \frac{1}{\hat{\beta}^{(i)}} Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)} \end{aligned} \quad (4)$$

$$\hat{\beta}^{(i)} = \left\| \left(Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)} \right) \right\|$$

$$\beta^{(i)} = {x^{(i)}}^T Q_s^{-1}(\alpha^{(i)}) R_s^{(i)} x^{(i)} \;\; and \;\; r^{(i)} = \frac{\beta^{(i)}}{\hat{\beta}^{(i)}}$$

choose the value s, that reduce the residual at every iteration where, residual at i^{th} iteration $= Q_s(\alpha^{(i)}) x^{(i)}$

Convergence Criteria

Lemma

$$\left|\frac{\beta^{(i)}}{\hat{\beta}^{(i)}}\right| \leq 1, \text{ where } \beta^{(i)} = {x^{(i)}}^{T} Q_s^{-1}(\alpha^{(i)}) R_s x^{(i)} \text{ and}$$

$$\hat{\beta}^{(i)} = \left\| \left(Q_s^{-1}(\alpha^{(i)}) R_s x^{(i)} \right) \right\|$$

Convergence Criteria

Theorem

$$\frac{\|Res^{(i+1)}\|}{\|Res^{(i)}\|} < 1 \text{ if } \tilde{t} = \left(\frac{\beta^{(i)}}{\hat{\beta}^{(i)}}\right)^2 s, \text{ and } 0 \le s \le 1.$$

Results of Numerical Experiments

Numerical experiments on comparison between *The Jacobi-Davidson Method* and *The Hybrid method*. In our experiments,
M = Identity matrix, and
C, K = arbitrary symmetric positive definite Toeplitz matrices.
Order of matrices are 500 and 800

Example 1

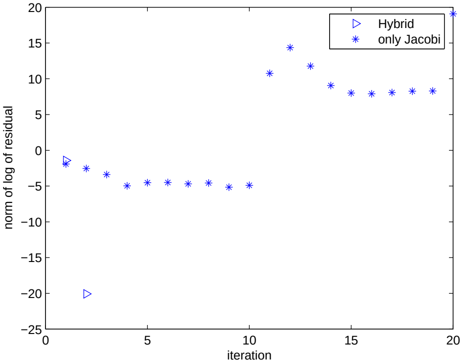
Matrix size(n) = 500, Interval [41.5 43.5]
The approximate initial eigenvalue determined by the Parametrized Newton method = 42.0307903.
Exact eigenvalues in interval [41.5 43.5] are:
41.59432, 41.62015, 41.93211, 43.00123.

TABLE 1: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 1

Methods	Residual	Iteration	Eigenvalue
Hybrid	$1.9e^{-9}$	2	41.93211
JD	No convergence	20	

Experiments

Figure: Norm of log of Residual *verses* Iteration



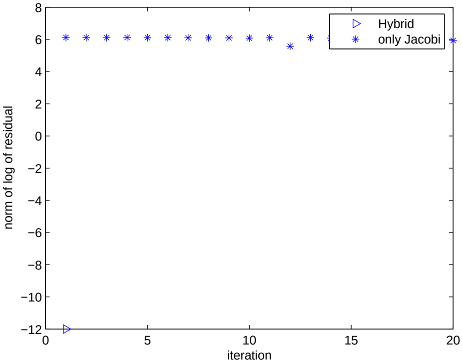
Example 2

Matrix size(n) = 800, Interval [70 73]
The approximate initial eigenvalue determined by the Parametrized Newton method = 71.063630.
Exact eigenvalues in interval [70 73] are:
71.488815, 72.89726.

TABLE 2: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 2

Methods	Residual	Iteration	Eigenvalue
Hybrid	$6.61e^{-6}$	1	71.488815
JD	No convergence	20	

Figure: Norm of log of Residual *verses* Iteration



Example 3

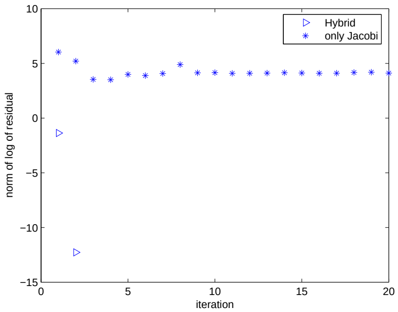
Matrix size(n) = 800, Interval [60 62]
The approximate initial eigenvalue determined by the Parametrized Newton method = 60.9958.
Exact eigenvalues in interval [60 62] are:
60.3894, 60.9596, 61.3442, 61.7969, 61.9379.

TABLE 3: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 3

Methods	Residual	Iteration	Eigenvalue
Hybrid	$5.0e^{-6}$	2	60.3894
JD	No convergence	20	

Experiments

Figure: Norm of log of Residual *verses* Iteration



Observation from the experiment

Numerical experimental results show that the hybrid method converges faster than the Jacobi-Davidson method alone; indeed, in some cases when the Jacobi-Davidson method did not converge at all, the new method worked quite well.

Future research

- The method is parametric in nature and rate of convergence depends upon the appropriate choice of the parameter. Studies on how to choose it properly to guarantee or accelerate the convergence is currently underway.
- Find efficient way to solve linear system in Modified Parametrized Newton's method.
- Find the convergence rate of Hybrid method.
- Find all the eigenvalues in the given interval.

Bibliography

1 KARABI DATTA, Y.P. HONG AND R. LEE, *A Parametrized Newton's Iteration for computing an eigenpair of a real symmetric matrix in an interval*, Computational Methods in Applied Mathematics, Volume 3 (2003) No, 4, pp. 517 - 535.

(2) G.L.G SLEIJPEN, G.L. BOOTEN AND D.R. FOKKEMA, AND H.A. VAN DER VORST, *Jacobi Davidson type methods for generalized eigenproblems and polynomial eigenproblems. BIT*, 1996, 36, pp. 595 - 633.

(3) Z. BAI, J. DEMMEL, J. DONGORRA, A. RUHE AND H.A. VAN DER VORST, *Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide*, SIAM, Philadelphia, 2000.

(4) B.N. DATTA, *Numerical Linear Algebra and Application*, Brooks / Cole Publishing Co., 1995.

(5) F. TISSEUR AND K. MEERBERGEN, *The quadratic eigenvalue problem*, SIAM REVIEW, Vol. 43, No. 2, 2001, pp. 235 - 286.

(6) G.L.G SLEIJPEN AND H. A. VAN DER VORST, *A JacobiŰDavidson iteration method for linear eigenvalue problems*, SIAM J. Matrix Anal. Appl. 17, 2 (1996), pp. 401 - 425.

(7) G.L.G SLEIJPEN AND H. A. VAN DER VORST, *The JacobiŰDavidson method for eigenvalue problems and its relation with accelerated inexact Newton schemes*, In Iterative Methods in Linear Algebra, II, S. D. Margenov and P. S. Vassilevski, eds., IMACS Ann. Comput. Appl. Math. 3, Baltzer, Basel, 1996, pp. 377 - 389.

(8) G.L.G. SLEIJPEN AND H. A. VAN DER VORST, *A Jacobi-Davidson iteration method for linear eigenvalue problems*, SIAM Review 42, 2 (2000), pp. 267 - 293.

(9) DANIIL R. SARKISSIAN, *Theory and Computations of Partial Eigenvalue and Eigenstructure Assignment Problems in Matrix Second-Order and Distributed-Parameter Systems*, DISSERTATION, Northern Illinois University, 2001.

(10) P. LANCASTER, *Lambda-Matrices and Vibrating systems*, Pergamon Press, Oxford, UK, 1966.

(11) Z. Bai AND Y. SU, *SOAR: A Second-Order Arnoldi Method for the Solution of the Quadratic Eigenvalue Problem*, SIAM J. Matrix Anal. Appl. 26, 3 (2005), pp. 640 - 659.

THANK YOU