Recent Developments on Quaternion Matrices

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$$1, \quad \mathbf{i}, \quad \mathbf{j}, \quad \mathbf{k}$$

$$q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

$$q = c_1 + c_2\mathbf{j}$$

$$x \sim pxp^{-1}$$

$$\mathbf{ij} \neq \mathbf{ji}$$

$$pq \neq qp$$

Quaternions do not commute!

[*Skew* field; Division algebra; Not a C^*-algebra]

P.M. Cohn: *Skew Fields*, Cambridge U. Press, 1995.

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Research Involving Quaternions

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A Critical Point:

What is a meaningful research for quaternions (matrices)?

Methods and *Results*

$$A = A_1 + A_2\mathbf{j}$$

$$\lambda A \neq A\lambda$$

$$\overline{AB} \neq \bar{A}\bar{B}$$

$$(AB)^t \neq B^t A^t$$

$$(AB)^* = B^* A^*$$

Note: Identities of complex matrices do not hold for quaternion matrices when $\bar{\cdot}$ OR $\cdot^t$ is applied. They hold when BOTH are applied.

- L.A. Wolf on *Similarity*, 1939
- H.C. Lee on *right eigenvalues*, 1949
- J.L. Brenner on *triangularization*, 1951
- R. Kippenhahn on *numerical range*, 1951
- N.A. Weigmann on *Jordan form*, 1955
- J. Jamison on *numerical range*, 1972
- M. Mehta on *det of quaternion matrices*, 1974
- Y.H. Au-Yeung on *convexity*, 1984
- R. Wood (1985), L. Huang and W. So (2001) on *left eigenvalues*
- A. Bunse-Gerstner, R. Byers, V. Mehrmann on *QR algorithm*, 1989
- W. So, R. Thompson and F. Zhang on *Numerical range*, 1994, 1996
- D.R. Farenick and B.A.F. Pidkowich on *spectral theorem*, 2003
- L. Rodman on *canonical forms for quaternion matrix pencils*, 2006
- F. Zhang 2 LAA papers, 1997 & 2007

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$$Ax = \lambda x \quad \text{Vs.} \quad Ax = x\lambda$$

left Vs. *right*

Q1: Existence?

Q2: How many?

Q3: Structure?

Right eigen: a lot has been done. Good: easy; Bad: (Singularity).

Left eigen: very little is known so far. Good:(Singularity); Bad: Not easy.

Wood: Bull. London Math. Soc. 1985

Example 1:

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}.$$

Then

$$\sigma_r(A) = \{1, -1\}$$

and

$$\sigma_l(A) = \{\lambda : \lambda = \alpha + \beta\mathbf{j} + \gamma\mathbf{k}, \}$$

where

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

- A is Hermitian, $1, -1$, $\mathbf{j}$, and $\mathbf{k}$ are left eigenvalues. Thus

$$\sigma_r(A) \subset \sigma_l(A).$$

- Left and right eigenvalues differ even for Hermitian matrices.

Example 2:

$$A = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{j} & 0 \end{bmatrix}.$$

Then

$$\sigma_l(A) = \left\{ \pm \frac{1}{\sqrt{2}} (\mathbf{i} + \mathbf{j}) \right\}$$

and

$$\sigma_r(A) = \{\lambda \in \mathbb{Q} : \ \lambda^4 + 1 = 0\}.$$

- No left eigenvalue is a right eigenvalue.
- Finite left eigen. infinite right eigen.
- Similar matrices have diff. left eigen.
- Matrix of diff. left eigen not diag.
- A and A^t may have diff. left eigen.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Why the 2×2 case is important:

- Toplitz-Hausdorff Theorem
- Upper bill

How to find eigen. of a 2-by-2 matrix?

Over $\mathbb{C}$: λ is an eigenvalue of A iff

$$\begin{aligned} Ax = \lambda x \quad &\Leftrightarrow \quad (A - \lambda I)x = 0 \\ &\Leftrightarrow \quad \det(A - \lambda I) = 0 \end{aligned}$$

Over $\mathbb{Q}$: Structure of left eigen.?

Theorem (Huang and So, LAA 2001)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $bc \neq 0$. *Then* l *is a left eigen. of* A *iff*

$$l = a + bx$$

where x *satisfies*

$$bx^2 + (a-d)x - c = 0.$$

Over $\mathbb{C}$: If λ is an eigen. of A, then

$$(\lambda - a)\,(\lambda - d) = b\,c \tag{1}$$

which yields

$$|\lambda - a|\;|\lambda - d| = |b|\;|c|. \tag{2}$$

Over $\mathbb{Q}$: If λ is LEFT, (2) holds, but (1) does not.

Theorem (Zhang, LAA 2007)

Let $A = (a_{ij})$ be an $n \times n$ quaternionic matrix and let $\lambda \in \mathbb{H}$ be a left or right eigenvalue of A. Then

$$|\lambda| \leq \max_i \sum_{j=1}^{n} |a_{ij}| := R$$

and

$$\rho_l(A), \rho_r(A) \leq \max_{\|x\|=1} \|Ax\|.$$

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We shall call

$$\{z \in \mathbb{Q} : \ |z - a_{ii}| \le R_i(A)\}$$

a *Geršgorin ball*.
It may contain no complex numbers!

Geršgorin theorem for

Left eigenvalues: OK.

Right eigenvalues: No!

Theorem (Geršgorin Theorem for Right Eigenvalues. Zhang, LAA 2007)

Let $A = (a_{ij})$ be an $n \times n$ matrices of quaternions. For every right eigenvalue λ of A there exists a quaternion α such that $\alpha^{-1}\lambda\alpha$ (which is also a right eigenvalue) is contained in the union of the Geršgorin balls $\{q \in \mathbb{Q} : \ |q - a_{ii}| \leq R_i(A)\}$, i.e.,

$$\{z^{-1}\lambda z : \ 0 \neq z \in \mathbb{Q}\} \cap \cup_{i=1}^{n} \{q \in \mathbb{Q} : \ |q - a_{ii}| \leq R_i(A)\} \neq \emptyset.$$

In particular, when λ is real, it is contained in a Gersgŏrin ball.

Theorem (Huang and So, LAA 2001)

If A is a 2×2 quaternion matrix and if A has finite (distinct) left eigenvalues, then A has at most 2 (distinct) left eigenvalues.

That is: if A is a 2×2 quaternion matrix, then $|\sigma_l(A)| = 1, 2$, or ∞.

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Theorem (Zhang, LAA2007)

Let A be a 2×2 quaternion matrix. If the two Gersgorin balls of A are disjoint, then A has two distinct left eigenvalues and each ball contains a left eigenvalue.

Two G-balls


Ball 1


Ball 2

F. O. Farid, Q.-W. Wang, and F. Zhang consider the localization of left eigenvalues of 2×2 quaternion matrix and the Gerśgorin balls.

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– *Thanks* –