Algebra Sense

What is "algebra"?

Most students answer that algebra is "solving for *x*."

If this is correct:

Why do so many students have so much difficulty with algebra, even to the extent of changing their career goals to avoid algebra?

Why does the National Council of Teachers of Mathematics propose algebra for **all** students in elementary school?

Why is the course **after Calculus** in college called Linear **Algebra**?

Why is the senior-level mathematics course called Modern **Algebra**?
How can the stuff in elementary school for all students and the stuff for college mathematics majors be the same course? How does this make sense?

Make Sense of Algebra

The thing that most people do not understand (including a lot of high school mathematics teachers) is that algebra is **not one thing**. In fact, it involves **four** different, but connected, ideas in mathematics.

1. First, at the earliest elementary school levels, we have the algebra of the **Study of Generalized Arithmetic --** that is **Looking for Patterns**

   An Example: In this pattern, 1 X 5; 2 X 5; 3 X 5; 4 X 5; …
   what comes after 4 X 5?
   what is the seventh entry in this pattern?
   what is the "n-th" entry?

   Did you have to multiply the exercise out to see the pattern, or could you do this by looking at the pattern of the exercise problems themselves? If you did not need the exercise answers to see the pattern (i.e., 5 then 10 then 15 then 20) then you are thinking *algebraically* – with *algebra sense*.

2. Second, at the elementary/middle grade levels, we have the algebra of the **Study of Procedures** – that is **Solving for an Unknown Variable**

   An Example: Find the value for x in the equation; $5x + 3 = 40$

   This is the type of mathematics most people associate with algebra.

3. Third, at the middle school/secondary grade levels, we have the algebra of the **Study of Relationships Among Quantities** – that is **Seeing How Variables Depend on Each Other**

   An Example: Graph the equation; $y = 2x + 3$

   Notice we usually are not trying to solve for x, but see how the relationship between the independent variable x affects the dependent variable y. Most of high school algebra is studying the relationships among quantities. But most high school (and now middle school) students taking algebra don't know this is the course objective.

4. Last, at the secondary/college levels, we have the algebra of the **Study of Structure** – that is **Proving Mathematical Ideas**

   An Example: Prove for the sides in a right-angle triangle, that $a^2 + b^2 = c^2$

   Almost all higher level college mathematics is based upon proving mathematical conjectures. Learning to make valid arguments that prove your conjecture is a very valued skill.

Using the Four Algebra(s)

An Example:
If the teacher writes on the board in two columns the following:

1 X 1 = 1	0 X 2 = 0
2 X 2 = 4	1 X 3 = 3
3 X 3 = 9	2 X 4 = 8
4 X 4 = 16	3 X 5 = 15
5 X 5 = 25	4 X 6 = 24

(a) What is the next line on the left side?

(b) What is the next line on the right side?

(c) If on the left side we see 15 X 15 = 125 what then will 14 X 16 = ?

Now extend these ideas:

(d) Multiply 149 X 151 mentally. (Hint, which side (left or right) will this be on? What's on the other side? How does this connect to the above 15 X 15 expression?)

(e) Factor 3,599 mentally. (Again, which side will this be on? What's on the other side? What would give you this answer on the other side?)

(f) Give an algebraic generalization of the above arithmetic pattern(s).

(g) Prove $(x - 1) \bullet (x + 1) = x^2 - 1$

Reflecting back on what we just did:

Which algebra did we do in (a and b)?	**Generalized Arithmetic**
Which algebra did we do in (c)?	**Relationships**
Which algebra did we do in (d and e)?	**Procedures**
Which algebra did we do in (f and g)?	**Structures**

Why Students Have Difficulty with Algebra

Talking algebra is like talking street slang. You may know the words being used, but you have to be familiar with the **context** to understand. Algebra uses a lot of slang and abbreviated expressions. In fact this gets us into many confusing situations.

There are only 10 numerals (0, 1, 2 … 9) that we use.
There are only 26 letters in the alphabet that we can use.
There are only 24 lowercase and 11 unique uppercase Greek symbols that we use.
There are approximately 60 common mathematical symbols (e.g., +) that we use.

With all the mathematics, at all the levels, that we need, we end up "reusing" many things to mean different ideas, based on the context when used.

a. Example: What does *x* mean in a high school algebra class?

Most people say it stands for a label, or a variable.
Elementary students usually say the multiplication sign or the Roman numeral for ten.
High school mathematics teachers will say the abscissa or horizontal axis in a graph.
Most students think, "I got the problem wrong" when they first see an *x* on their paper!
Who is correct? You cannot tell unless you know the context *x* is being used.

b. Example: What does 8*x* mean?

Most people will say it means 8 "times" *x*, it implies multiplication by the variable *x*.
Now, if *x* is a variable, could *x* be 7? Substituting, would 87 imply 8 "times" 7?
(This situation is called "concatenation.")

c. The difference between a label and a variable in word problems:

Write a mathematical equation for the following using s and p:

"There are six times as many students as professors"

Most people will write **6 • s = p**
Substitute 30 in for students. Does having 180 professors seem correct? The s and the p are labels, not variables.

The problem is when students try to do a **direct translation** of the English into mathematical expressions. It is similar to taking a foreign language and doing word-by-word translation. It will not make sense. So it also is with the language of mathematics.

There is a simple method to help students write mathematically correct expressions. It is to first rewrite the English sentence into variables and mathematical operations.

Now first rewrite the sentence in English **"There are six times as many students as professors"** using:

"the number of students" and
"the number of professors" in the original sentence.

Now write a mathematical equation using s and p for this sentence. Many people will have something like, **"The number of students is equal to six times the number of professor."** If we now do **a direct translation** of the second sentence, we get: **s = 6 • p**
This is a correct equation. The s and the p now are **variables**.

Reflection

Thus the "algebra" we do in early elementary school is **Generalized Arithmetic, Looking for Patterns.** The "algebra" of middle school is **Procedures**, **Solving for an Unknown Variable.** The "algebra" of high school is **Relationships Among Quantities, Seeing How Variables Depend on Each Other.** And lastly, the "algebra" of college is **Structure**, **Proving Mathematical Ideas.** All are interwoven.

Extending Further

Look at this example:

3 apples and 2 bananas cost 13 cents
2 apples and 1 banana cost 8 cents

(i) What is the price of 5 apples and 3 bananas?

(ii) What is the price of 6 apples and 4 bananas?

(iii) What is the price of 8 apples and 5 bananas?

If we write these two sentences as equations we have:

$$3a + 2b = 13¢$$
$$2a + 1b = 8¢$$

And in Linear Algebra we write these two equations in matrix form of:

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 8 \end{bmatrix}$$

So in Linear Algebra we can take complete "lines of mathematics" and do operations (add, subtract, multiply, divide) to them. To solve (i) we can add line one to line two and get:
$5a + 3b = 21¢$

References:

Zalman Usiskin "Conceptions of School Algebra and Uses of Variables" in Louise Chalouh & Nicolas Herscovics "Teaching Algebraic Expressions in a Meaningful Way' in *The Ideas of Algebra, K-12: 1988 Yearbook*. National Council of Teachers of Mathematics: Reston, VA.

Mark Saul (1997). Algebra, Technology and a Remark of I.M. Gelford. In The Nature and Role of Algebra in the K-14 Curriculum. National Academy Press, Washington, DC.

Alan Zollman (1997). "The Teaching of Algebra" Northern Illinois University. DeKalb, IL.