MATH 430
FINAL
Fall 2006

Prof. Harris

Name ____________________

ID ____________________

1. Prove that for any $C \in \mathbb{R}$, the equation

$$x^3 + 3x + C = 0$$

has at most one solution in the interval $[-1, 1]$.

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2. Use L'Hôpital's rule to evaluate

$$\lim_{x \to 0} \frac{x^2 \sin x}{\sin x - x \cos x}$$

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3. Use the definition of the derivative to find the derivative of $f(x) = 2\sqrt{x} + 1$ at $x_0 \in (0, \infty)$. Does $f'(0)$ exist? Explain.

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4. Prove, using ε and δ that $f(x) = x^2$ is continuous at any $x \in [10, 21)$. Is it uniformly continuous on this interval? State any theorems that you use.

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5. Prove that the set $S = \left\{ \left(\frac{x}{2}, n^2 \right) \middle| x \in \mathbb{Q},\ n \in J \right\}$ is countable.

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6. Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Let $x = \inf\{S\}$. If $x \notin S$, prove that x is an accumulation point of S.

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7. The set $(0, 1]$ is not compact. Find an infinite open cover with no finite subcover.

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8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in \mathbb{R}$, prove that $F(x) = f(x)+8$ is increasing. What, if any, extra conditions would need to be imposed for $G(x) = (f(x))^2$ to be increasing?

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9. If $c > 1$ prove that $\{\sqrt[n]{c}\}$ is convergent. What is it convergent to?

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10. Prove directly that $f(x) = x$ is Riemann integrable on $[0, 1]$.

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11. If $a_n,\ b_n > 0$ for all $n \in J$, $\{a_n^2\}$ is convergent and $\{b_n^2\}$ is Cauchy, prove that $\{a_n b_n\}$ is convergent. State any theorems that you use.

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12. Must every bounded sequence in $\mathbb{R}$ be convergent? Must every bounded sequence contain a convergent subsequence? Is every convergent sequence bounded? Give examples.

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13. Show that

$$2^{2^x} = 6x \text{ for some } x \in (0,1)$$

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14. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) with $f(a) = f(b) = 0$. Prove that for each $k \in \mathbb{R}$ there is a $c \in (a,b)$ with $f'(c) = kf(c)$. Hence, or otherwise, show that it is always possible to solve the equation $\cos x = k \sin x$ for $x \in (0,\pi)$ for all $k \in \mathbb{R}$

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15. $f : [1, 2] \to [1, 2]$ is continuous with $f([1, 2]) = [1, 2]$. Prove that there exists $c \in (1, 2)$ with $f(c) = c$

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