MATH 430
FINAL
Fall 2011

Name ____________________

ID ____________________

1. Prove that for any $C \in \mathbb{R}$, the equation

$$x^3 + 3x + C = 0$$

has <u>at most</u> one solution in the interval $[-1, 1]$.

15

2. Use L'Hôpital's rule to evaluate

$$\lim_{x \to 0} \frac{x^2 \sin x}{\sin x - x \cos x}$$

15

3. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x} + 1$ at $x_0 \in (0, \infty)$. Does $f'(0)$ exist? Explain.

15

4. Prove that $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ is not Riemann Integrable on $[0, 1]$.

15

5. Prove that the set $S = \left\{ \left(m, n, \frac{p}{q} \right) \middle| m, n, p, q, \in J \right\}$ is countable.

10

6. Let S be a non-empty infinite subset of $\mathbb{R}$ which is bounded above. Let $x = \sup\{S\}$. If $x \notin S$, prove that x is an accumulation point of S.

10

7. The set $(0, 1]$ is not compact. Find an infinite open cover with no finite subcover.

10

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $f'(x) > 0$ for all $x \in \mathbb{R}$, prove that $F(x) = f(x) + 7$ is increasing. If $G(x) = (f(x))^2$ is $G(x)$ necessarily an increasing function? If you think the answer is "Yes", prove it; if you think the answer is "No", find extra conditions that must be imposed on f and prove your conjecture.

15

9. If $c > 1$ prove that $\{\sqrt[n]{c}\}$ is convergent. What is it convergent to?

10

10. Prove, using the definition, that $f(x) = x^2$ is Riemann Integrable on $[0, 1]$.

10

11. If $a_n,\ b_n > 0$ for all $n \in J$, $\{a_n^2\}$ and $\{b_n^2\}$ are both Cauchy, prove that $\{a_n b_n\}$ is convergent.

15

12. Is every bounded sequence in $\mathbb{R}$ be convergent? Is every convergent sequence bounded? Give counter examples or proofs.

15

13. Show that

$$2^{2^x} = 6x \text{ for some } x \in (0,1)$$

15

14. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) with $f(a) = f(b) = 0$. By considering, $e^{-kx} fx$ show that for each $k \in \mathbb{R}$ there is a $c \in (a,b)$ with $f'(c) = kf(c)$. Deduce that, for all $k \in \mathbb{R}$, it is always possible to solve the equation $\cos x = k \sin x$ for $x \in (0,\pi)$.

10

15. If $f : [1, 2] \to [1, 2]$ is continuous with the range of f being $[1, 2]$; prove that there exists $c \in (1, 2)$ with $f(c) = c$

20