Math 430 Final Exam
Spring 2000

Name (print): ____________________________

Instructions: Do 6/9 problems only. When you are finished with the exam, list below the problems you want graded.

1. #
2. #
3. #
4. #
5. #
6. #

1. Let $A = \{x \in R | p(x) = 0\}$ where p is a polynomial with integer coefficients. Prove that A is a countable set.

2. (a) State the Bolzano-Weierstrass Theorem.
   (b) Define what it means for $\{a_n\}$ to be Cauchy.
   (c) Use Bolzano-Weierstrass to prove that every Cauchy sequence of real numbers is convergent.

3. (a) Prove that

   i. $\{a_n\}$ is decreasing
      and

   ii. $\{a_n\}$ is bounded from below implies that $\{a_n\}$ is convergent.

(b) Show that the sequence

$$s_1 = \sqrt{2},\ s_n = \sqrt{2 + \sqrt{s_{n-1}}} \text{ for } n = 2, 3, \cdots$$

is convergent.

4. (a) Let $g : (0, 1) \to R$ be given by $g(x) = \dfrac{\sqrt{1+x}-1}{x}$. Find $L = \lim_{x\to 0} g(x)$ and prove, rigorously, that the limit is L

(b) Define $f(x) = \dfrac{\sqrt{x}}{1-\sqrt{x}}$ for $x > 1$. Prove that f has a limit at infinity and find it.

5. Let $f : [0, 2] \to R$ be monotone increasing.

   (a) Prove that $\lim_{x \to 2^-} f(x)$ exists.

   (b) Prove that $\lim_{x \to 0^+} f(x)$ exists.

6. Prove directly from the definitions that

(a) $f(x) = \dfrac{2}{x-3}$ is uniformly continuous on $[4, 5)$.

(b) $f(x) = \dfrac{1}{x} + 1$ is not uniformly continuous on $(0, 1]$.

7. Let $f : E \to R$ be continuous with E compact. Assume that f is also 1-1. Prove that

$$f^{-1} : f(E) \to E$$

is continuous.

8. (a) Use the mean-value theorem to prove that for $h > 0$ and $p < 1 : (1+h)^p < ph+1$ while for $p > 1$. $(1+h)^p > ph+1$

(b) Explain why the equation $x^3 - 3x + b = 0$ has at most one root in the interval $[-1, 1]$.

9. Let

$$f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0. & x = 0 \end{cases}$$

Show that f is differentiable for all $x \in R$ and f' is continuous at $x = 0$, but that f' is not differentiable at $x = 0$.