Name ______________________________

Math 431/531 Final Exam Spring 2010

1. Given the definitions of the following:

   (i) A metric space $(X,\ d)$.

   (ii) The pointwise convergence and the uniform convergence of a sequence $\{f_n\}$ to f where $f,\ f_n,\ n = 1, 2, 3, 4, \ldots$ are functions from a metric space $(X_1,\ d_1)$ into another metric space $(X_2,\ d_2)$.

   (iii) Given $f|D \subset \mathbb{R}^n \to \mathbb{R}^m$ and let x_0 be an interior point of D, what is $f'(x_0)$, the total derivative of f at x_0?

   (iv) Given a metric space $(X,\ d)$, what is a strict contraction mapping on X?

2. Let $C([a,\ b])$ be the space of all the functions which are continuous on $[a,\ b]$. We define

$$d(f,\ g) = \int_a^b |f(x) - g(x)|\ dx \qquad \forall f,\ g \in C([a,\ b])$$

   Prove that $C([a,\ b])$ equipped with d is a metric space. Is the space complete though? Explain briefly.

3. Find out the interval of convergence for the following. In particular, investigate the convergence at the end points.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n} \left(\frac{x-1}{x+1}\right)^n$

   (ii) $\displaystyle\sum_{n=1}^{\infty} \left[1 - \left(\frac{x+2}{n}\right)\right]^{n^2}$

4. Given $f|[0,\ 1] \to \mathbb{R}$ be continuous on $[0,\ 1]$, suppose further that f is differentiable in $(0,\ 1)$ with $|f'(x)| \le k \quad \forall x \in (0,\ 1)$ where k is positive constant strictly between 0 and 1. Prove that the equation $x - f(x) = 0$ has exactly one solution in $[0,\ 1]$.

5. Consider the sequence of functions $\{f_n\}$ defined by

$$f_n(x) = \frac{x^n}{1+\sqrt{x}}, \quad x \in [0,\ 1]$$

(i) Find $f(x) = \lim_{n\to\infty} \dfrac{x^n}{1+\sqrt{x}}$

(ii) Determine if the convergence of f_n to f is pointwise or uniform.

(iii) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$. (Hint: use dominated convergence theorem.)

6. Consider a surface whose equation is given in the implicit form $F(x,\ y,\ z) = 0$. Let $P(x_0,\ y_0,\ z_0)$ be a point on the surface such that each variable could be expressed as a function of the other variables locally at P, i.e., in a small neighborhood of P, the surface has explicit representations in the forms $z = h(x,\ y)$, $x = g(y,\ z)$, and $y = g(x,\ z)$, respectively.

   (i) Prove that $\dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} = -1$ at P.

   (ii) Show that this is so everywhere on the surface, $x^3 + y^3 + z^3 + 4x + 3y + 7z = 8$.

7. Given $f(x,\ y) = \begin{bmatrix} x^3 + y^3 \\ xy \\ y^2 - x^2 \end{bmatrix}$, find $f'(1,\ 1)$.

   and verify that indeed the $f'(1,\ 1)$ you got satisfies the condition in the definition of the total derivative.

8. (i) Derive a power series representation for $\ln(1+x)$ for $-1 < x < 1$ using the integral formula.

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt$$

   State clearly what is the reason for restricting x to be in $(-1,\ 1)$.

   (ii) Using your result in (i), obtain an infinite series representation for $\ln 2$.

9. **Bonus.** Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k}$ is convergent.