SOLUTIONS TO THE REVIEW PROBLEMS

1. (a) What are the possibilities for the order of an element of $\mathbf{Z}_{13}^{\times}$? Explain your answer.

   Solution: The group $\mathbf{Z}_{13}^{\times}$ has order 12, and the order of any element must be a divisor of 12, so the possible orders are 1, 2, 3, 4, 6, and 12.

   (b) Show that $\mathbf{Z}_{13}^{\times}$ is a cyclic group.

   Solution: The first element to try is [2], and we have $2^2 = 4$, $2^3 = 8$, $2^4 = 16 \equiv 3$, $2^5 \equiv 2 \cdot 2^4 \equiv 6$, and $2^6 \equiv 2 \cdot 2^5 \equiv 12$, so the order of [2] is greater than 6. By part (a) it must be 12, and thus [2] is a generator for $\mathbf{Z}_{13}^{\times}$. We could also write this as $\mathbf{Z}_{13}^{\times} = \langle [2]_{13} \rangle$.

2. Find all subgroups of $\mathbf{Z}_{11}^{\times}$, and give the lattice diagram which shows the inclusions between them.

   Solution: First check for cyclic subgroups, in shorthand notation: $2^2 = 4$, $2^3 = 8$, $2^4 = 5$, $2^5 = 10$, $2^6 = 9$, $2^7 = 7$, $2^8 = 3$, $2^9 = 6$, $2^{10} = 1$. This shows that $\mathbf{Z}_{11}^{\times}$ is cyclic, so the subgroups are as follows, in addition to $\mathbf{Z}_{11}^{\times}$ and $\{[1]\}$: $\left\langle [2]^2 \right\rangle = \{[1], [2]^2, [2]^4, [2]^6, [2]^8\} = \{[1], [4], [5], [9], [3]\}$ and $\left\langle [2]^5 \right\rangle = \{[1], [2]^5\} = \{[1], [10]\}$ The lattice diagram forms a diamond.

3. Let G be the subgroup of $GL_3(\mathbf{R})$ consisting of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ such that } a, b \in \mathbf{R}\,.$$

   Show that G is a subgroup of $GL_3(\mathbf{R})$.

   Solution: We have $\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+c & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the closure property holds. The identity matrix belongs to the set, and $\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the set is closed under taking inverses.

4. Show that the group G in the previous problem is isomorphic to the direct product $\mathbf{R} \times \mathbf{R}$.

   Solution: Define $\phi : G \to \mathbf{R} \times \mathbf{R}$ by $\phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = (a, b)$. This is one-to-one and onto because it has an inverse function $\theta : \mathbf{R} \times \mathbf{R} \to G$ defined by $\theta((a,b)) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Finally, ϕ preserves the respective operations since $\phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 1 & a+c & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) =$

$(a+c, b+d) = (a,b) + (c,d) = \phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$.

5. List the cosets of the cyclic subgroup $\langle 9 \rangle$ in $\mathbf{Z}_{20}^{\times}$. Is $\mathbf{Z}_{20}^{\times} / \langle 9 \rangle$ cyclic?

Solution: $\mathbf{Z}_{20}^{\times} = \{\pm 1, \pm 3, \pm 7, \pm 9\}$.

$\langle 9 \rangle = \{1, 9\}$ $\quad (-1)\langle 9 \rangle = \{-1, -9\}$ $\quad 3\langle 9 \rangle = \{3, 7\}$ $\quad (-3)\langle 9 \rangle = \{-3, -7\}$

Since $x^2 \in \langle 9 \rangle$, for each element x of $\mathbf{Z}_{20}^{\times}$, the factor group is not cyclic.

6. Let G be the subgroup of $GL_2(\mathbf{R})$ consisting of all matrices of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$, and let N be the subset of all matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$.

(a) Show that N is a subgroup of G, and that N is normal in G.

Solution: The set N is nonempty since it contains the identity matrix, and it is a subgroup since $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} =$

$\begin{bmatrix} 1 & b-c \\ 0 & 1 \end{bmatrix}$. N is normal in G since $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}^{-1} =$

$\begin{bmatrix} m & mc+b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/m & -b/m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & mc \\ 0 & 1 \end{bmatrix} \in N$.

(b) Show that G/N is isomorphic to the multiplicative group $\mathbf{R}^{\times}$.

Solution: Define $\phi : G \to \mathbf{R}^{\times}$ by $\phi\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\right) = m$. Then we have $\phi\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n & c \\ 0 & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} mn & mc+b \\ 0 & 1 \end{bmatrix}\right) = mn =$

$\phi\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\right)\phi\left(\begin{bmatrix} n & c \\ 0 & 1 \end{bmatrix}\right)$. Since m can be any nonzero real number, ϕ maps G onto $\mathbf{R}^{\times}$, and $\phi\left(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\right) = 1$ if and only if $m = 1$, so $N = \ker(\phi)$. The fundamental homomorphism theorem implies that $G/N \cong \mathbf{R}^{\times}$.

Note that this part of the proof covers part (a), since once you have determined the kernel, it is always a normal subgroup. Thus parts (a) and (b) can be proved at the same time, using the argument given for part (b).

7. Assume that the dihedral group D_4 is given as $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where $a^4 = e$, $b^2 = e$, and $ba = a^3b$. Let N be the subgroup $\langle a^2 \rangle = \{e, a^2\}$.

   (a) Show by a direct computation that N is a normal subgroup of D_4.

   Solution: We have $a^i a^2 a^{-i} = a^2$ and $(a^i b)a^2(a^i b)^{-1} = a^i a^{-2} b a^i b =$

   $a^i a^{-2} a^{-i} b^2 = a^{-2} = a^2$, for all i, which implies that N is normal.

   (b) Is the factor group D_4/N a cyclic group?

   Solution: The cosets of N are

   $N = \{e, a^2\}$, $Na = \{a, a^3\}$, $Nb = \{b, a^2b\}$, and $Nab = \{ab, a^3b\}$.

   Since b and ab have order 2, and $a^2 \in N$, we see that each element in the factor group has order 2, so G/N is not cyclic.

8. Let $G = D_8$, and let $N = \{e, a^2, a^4, a^6\}$.

   (a) List all left cosets and all right cosets of N, and verify that N is a normal subgroup of G.

   Solution: The right cosets of N are

   $N = \{e, a^2, a^4, a^6\}$, $\quad Na = \{a, a^3, a^5, a^7\}$,

   $Nb = \{b, a^2b, a^4b, a^6b\}$, $\quad Nab = \{ab, a^3b, a^5b, a^7b\}$.

   The left cosets of N are more trouble to compute, but we get

   $N = \{e, a^2, a^4, a^6\}$, $\quad aN = \{a, a^3, a^5, a^7\}$,

   $bN = \{b, a^6b, a^4b, a^2b\}$, $\quad abN = \{ab, a^7b, a^5b, a^3b\}$.

   The fact that the left and right cosets of N coincide shows that N is normal.

   (b) Show that G/N has order 4, but is not cyclic.

   Solution: It is clear that there are 4 cosets. We have $NaNa = Na^2 = N$, $NbNb = Ne = N$, and $NabNab = Ne = N$, so each coset has order 2.