MATH 211 C EXAM III NAME SOLUTIONS

1. (10 pts; see p 243 Example 6) Find the absolute max and min values of $f(x) = 5x + \dfrac{35}{x}$ on $[1,5]$.

$f'(x) = 5 - 35x^{-2} = 5 - \dfrac{35}{x^2}$ and setting $f'(x) = 0$ gives $x = \pm\sqrt{7}$. The only critical point in $[1,5]$ is $x = \sqrt{7}$. We have $f(1) = 40$, $f(5) = 32$, and $f(\sqrt{7}) = 5\sqrt{7} + \dfrac{35}{\sqrt{7}} = 5\sqrt{7} + \dfrac{35\sqrt{7}}{\sqrt{7}\sqrt{7}} = 5\sqrt{7} + 5\sqrt{7} = 10\sqrt{7} < 10 \cdot 3 = 30$, so $f(\sqrt{7}) = 10\sqrt{7}$ is the absolute minimum on $[1,5]$, while $f(1) = 40$ is the absolute maximum.

2. (15 pts; p 254 Example 4) A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit must be $p = 1000 - x$. The total cost of producing x units is given by $C(x) = 3000 + 20x$. How many units must the company produce and sell in order to maximize profit?

$P(x) = R(x) - C(x) = x(1000 - x) - (3000 - 20x) = -x^2 + 980x - 3000$. Setting $P'(x) = 0$ gives the critical point $x = 490$, and this *does* produce a maximum since $P''(x) = -2$ means that the graph is concave down for all x.

3. (5 pts for each part) Find the derivative of each of these functions.

(a) $f(x) = e^{-x^2+7x}$ $\quad f'(x) = e^{-x^2+7x}(-2x+7)$

(b) $f(x) = \ln(5x^2 - 7)$ $\quad f'(x) = \dfrac{10x}{5x^2 - 7}$

(c) $f(x) = \dfrac{e^{2x} - 1}{e^x + 1}$ $\quad f'(x) = \dfrac{(2e^{2x})(e^x + 1) - (e^{2x} - 1)(e^x)}{(e^x + 1)^2}$

(d) $f(x) = e^{\sqrt{x}} + \sqrt{e^x} = e^{x^{1/2}} + (e^x)^{1/2} = e^{x^{1/2}} + e^{\frac{1}{2}x}$ $\quad f'(x) = e^{x^{1/2}}\left(\dfrac{1}{2}x^{-1/2}\right) + e^{\frac{1}{2}x}\left(\dfrac{1}{2}\right)$

(e) $f(x) = \ln\left(\dfrac{e^x\sqrt{x^4+1}}{(x^2+1)^3}\right) = \ln e^x + \ln\left((x^4+1)^{1/2}\right) - \ln\left((x^2+1)^3\right) = x + \dfrac{1}{2}\ln(x^4+1) - 3\ln(x^2+1)$

$$f'(x) = 1 + \left(\frac{1}{2}\right)\left(\frac{4x^3}{x^4+1}\right) - 3\left(\frac{2x}{x^2+1}\right) = 1 + \frac{2x^3}{x^4+1} - \frac{6x}{x^2+1}$$

4. (15 pts) The size of a certain population at time t is given by $P(t)$. The function $P(t)$ has these properties: $P'(t) = .08P(t)$ and $P(0) = 3{,}000$. Find the formula for $P(t)$, and then find $P(10)$.

We're given $k = .08$, so $P(t) = 3000e^{.08t}$, and $P(10) = 3000e^{(.08)(10)} = 3000e^{.8} = 3000 \times 2.226 = 6{,}678$.

5. (15 pts) A physics professor notices that attendance in her class seems to be decreasing exponentially. After starting with 100 students, there are 90 attending after 4 weeks. How many would she expect to be attending after 8 weeks?

$P(t) = 100e^{kt}$, and $90 = 100e^{4k}$, so $.9 = e^{4k}$ and then $k = \dfrac{1}{4}\ln(.9)$. To find $P(8)$, substitute $t = 8$ into the formula. This gives $P(8) = 100e^{8k} = 100e^{2\ln(.9)} = 100(e^{\ln(.9)})^2 = 100(.9)^2 = 100(.81) = 81$.

6. (20 pts; p 263 #33) A rectangular box with a volume of 350 ft^3 is to be constructed with a square base and top. The cost per square foot for the bottom is 15 cents, for the top is 10 cents, and for the sides is 2.5 cents. What dimensions will minimize the cost?

Total cost: $C(x) = 0.15x^2 + 0.1x^2 + 0.1xy = 0.25x^2 + 0.1xy$. We have $x^2y = 350$, so $C(x) = 0.25x^2 + 0.1xy = 0.25x^2 + 0.1x\left(\dfrac{350}{x^2}\right) = 0.25x^2 + \dfrac{35}{x}$. Setting $C'(x) = .5x - \dfrac{35}{x^2} = 0$, we get $x^3 = 70$, so $x = \sqrt[3]{70}$. This does minimize cost since $C''(x) = .5 + \dfrac{70}{x^3}$ is positive for $x = \sqrt[3]{70}$.

Answer: $x = \sqrt[3]{70}$ and $y = \dfrac{350}{(\sqrt[3]{70})^2} = \dfrac{350\sqrt[3]{70}}{(\sqrt[3]{70})^2\sqrt[3]{70}} = 5\sqrt[3]{70}$.

EXTRA CREDIT: Find the maximum and minimum values of $f(x) = x^4e^{-x}$ on $[0,10]$.

$f'(x) = 4x^3e^{-x} - x^4e^{-x} = x^3(4-x)e^{-x}$, and setting $f'(x) = 0$ gives $x = 0$ or $x = 4$, since $e^{-x} \neq 0$. Because $f'(x)$ changes from positive to negative at $x = 4$, it gives a relative maximum. We have $f(0) = 0$ and $f(10) > 0$, so on the interval $[0,10]$ the absolute minimum is $f(0) = 0$ and the absolute maximum is $f(4) = 256/e^4$.