MATH 240 **EXAM II SOLUTIONS** 10/22/2002

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1. (25 pts) Let A be the following matrix. $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 1 & -5 & -1 & 8 \\ 3 & 1 & -1 & -2 \end{bmatrix}$

(a) (8 pts) Find the reduced row echelon form of A.
(b) (4 pts) Find the rank and nullity of A.
(c) (4 pts) Find a basis for the row space of A.
(d) (4 pts) Find a basis for the column space of A.
(e) (5 pts) Find a basis for the nullspace of A.

Comment: The matrix A is almost the same as the matrix on the first exam, from Example 12 on page 54 of the text (I added a linearly dependent row: row 3 = row 2 - row 1).

$$A \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 7 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -10 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus A has rank 3 and nullity $4 - 3 = 1$; the rows of the reduced matrix form a basis for the row space; the first three columns of A form a basis for the column space; setting $x_4 = 1$ gives $x_3 = -3$, $x_2 = -1$, and $x_1 = -1$, and produces a single vector that is a basis for the nullspace of A.

2. (20 pts) Let M_{22} be the vector space of all 2×2 matrices, and let $Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(a) (10 pts) Let W be the set of all matrices A in M_{22} with $AQ = 0$. Show that W is a subspace of M_{22}.

If c_1 and c_2 are any scalars, and A_1, A_2 belong to W, then $(c_1A_1 + c_2A_2)Q = c_1A_1Q + c_2A_2Q = 0 + 0 = 0$, and this shows that W is a subspace. OR: First find the form of a typical matrix in W, as below.

(b) (10 pts) Find a basis for W, and find the dimension of W.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that $AQ = 0$. Then $AQ = \begin{bmatrix} a - b & -a + b \\ c - d & -c + d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ shows that $a = b$ and $c = d$. Since $\begin{bmatrix} a & a \\ d & d \end{bmatrix} = a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, these two matrices span W, and it is easy to see that they are not linearly dependent, so they form a basis. Conclusion: $\dim(W) = 2$.

3. (15 pts) Let $S = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}$ and $T = \{(1,-1,0),\ (0,1,-1),\ (0,0,1)\}$ be ordered bases for $\mathbf{R}_3$. Let $\mathbf{v} = (3,2,1)$.

(a) (4 pts) Find the coordinate vector $[\mathbf{v}]_T$ of $\mathbf{v}$ with respect to the basis T.

By trial and error, $(3,2,1) = 3(1,-1,0) + 5(0,1,-1) + 6(0,0,1)$, or solve a system of equations.

(b) (8 pts) Find the transition matrix $P_{S \leftarrow T}$. *Note that the vectors go in as columns.*

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

(c) (3 pts) Use $P_{S \leftarrow T}$ to find the coordinate vector $[\mathbf{v}]_S$ of $\mathbf{v}$ with respect to the basis S.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ *Check that this is correct by adding the vectors in S.*

4. (15 pts) Let M_{22} be the vector space of all 2×2 matrices. For A, B in M_{22}, define an inner product by $(A, B) = \operatorname{tr}(B^T A)$. (*Hint:* $\operatorname{tr}(A)$ denotes the trace of A, which is the sum of entries on the main diagonal.)

(a) (7 pts) Check that $(A, B) = (B, A)$ for all A, B in M_{22}.

Since $\operatorname{tr}(X) = \operatorname{tr}(X^T)$, we have $(A, B) = \operatorname{tr}(B^T A) = \operatorname{tr}((B^T A)^T) = \operatorname{tr}(A^T B) = (B, A)$.

(b) (8 pts) For any 2×2 matrix A, check that $(A, A) = 0$ if and only if $A = 0$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ then $(A, B) = \operatorname{tr}(B^T A) = \operatorname{tr}\left(\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right)$
$= \operatorname{tr}\left(\begin{bmatrix} b_{11}a_{11} + b_{21}a_{21} & * \\ * & b_{12}a_{12} + b_{22}a_{22} \end{bmatrix} \right) = b_{11}a_{11} + b_{21}a_{21} + b_{12}a_{12} + b_{22}a_{22}$.

Note: this formula can be used to prove part (a). It shows that this inner product for 2×2 matrices behaves exactly like the ordinary dot product in $\mathbf{R}^4$.

In particular, $(A, A) = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2$, and a sum of squares is zero if and only if each of the terms is zero. Therefore $(A, A) = 0$ if and only if $A = 0$.

5. (15 pts) Let W be the subspace of R_4 spanned by the vectors $(1, -1, 1, 1)$ and $(1, 0, 2, 1)$. Use the Gram-Schmidt process to find an orthonormal basis for W.

Let $\mathbf{u}_1 = (1, -1, 1, 1)$ and $\mathbf{u}_2 = (1, 0, 2, 1)$.

Then $\mathbf{v_1} = (1, -1, 1, 1)$, and $\mathbf{v}_2$ is $\mathbf{u}_2$ minus its projection onto $\mathbf{v}_1$.

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 0, 2, 1) - \frac{(1 + 0 + 2 + 1)}{(1 + 1 + 1 + 1)}(1, -1, 1, 1) = (1, 0, 2, 1) - (1, -1, 1, 1) = (0, 1, 1, 0).$$

Finally, divide each vector by its length: $\mathbf{w}_1 = \frac{1}{2}(1, -1, 1, 1)$ and $\mathbf{w}_2 = \frac{1}{\sqrt{2}}(0, 1, 1, 0)$.

6. (10 pts) Answer EITHER part **A** OR part **B**.

A. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V, show that $(\mathbf{u}, \mathbf{v}) = \frac{1}{4}||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4}||\mathbf{u} - \mathbf{v}||^2$.

Proof:
$||\mathbf{u} + \mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v})$
$= (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) - ((\mathbf{u}, \mathbf{u}) - 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v})) = 4(\mathbf{u}, \mathbf{v})$
Now divide by 4.

Remember that $||\mathbf{x}|| = \sqrt{(\mathbf{x}, \mathbf{x})}$*, or more generally that* $(\mathbf{u}, \mathbf{v}) = \cos\theta\, ||\mathbf{u}||\, ||\mathbf{v}||$*, so* $||\mathbf{x}||^2 = (\mathbf{x}, \mathbf{x})$.

B. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal set of nonzero vectors in an inner product space V. Show that S is a linearly independent set.

See the proof of Theorem 3.4 on page 210.

Grades:

Class average: 73; Standard deviation: 15.3.
Grading scale: 85–100 A (8); 75–84 B (4); 60–74 C (10); 50–59 D (3); 40–49 F (2)

Chapter 4 has new material, so it is a good place for a new start. Doing the suggested homework problems is very important. If you have questions, be sure you ask them in class or come to my office to discuss them. I have office hours MTF 1-2, but I'm usually in my office after class, often until 6 or 7. You can also ask questions via email, if you don't have a chance to stop by my office.