

If $L : V \rightarrow W$ is a linear transformation, and S is basis for a V , while T is a basis for W , then we will use $M_{T \leftarrow S}(L)$ to denote the matrix of L with respect to the bases S and T .

1. (p 290 #8) Define $L : M_{22} \rightarrow M_{22}$ by $L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$, for A in M_{22} . Use the ordered bases $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

(a) Find the matrix representation of L with respect to S .

To find $M_{S \leftarrow S}(L)$, substitute each of the basis vectors in S into L ; then find the coordinates relative to S .

$$\begin{aligned} L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} & L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \\ L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} & L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

The standard coordinates of these matrices go in as columns.
$$M_{S \leftarrow S}(L) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

(b) Find the matrix representation of L with respect to T .

To find $M_{T \leftarrow T}(L)$, substitute each of the basis vectors in T into L ; then find the coordinates relative to T .

$$\begin{aligned} L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \\ L\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 7 & 0 \end{bmatrix} & L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

We need to form the matrix $[T \mid L(T)]$ and row reduce to get $[I \mid M_{T \leftarrow T}(L)]$.

$$\begin{aligned} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 7 & 0 \\ 1 & 0 & 0 & 0 & 4 & 3 & 0 & 3 \end{array} \right] &\rightsquigarrow & \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 7 & 0 \\ 0 & -1 & -1 & 0 & 3 & 2 & -3 & 3 \end{array} \right] \\ &\rightsquigarrow & \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 7 & 0 \\ 0 & 0 & -1 & 1 & 5 & 3 & -3 & 4 \end{array} \right] &\rightsquigarrow & \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -4 & -3 & -4 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 8 & 6 & 4 & 4 \end{array} \right] \\ &\rightsquigarrow & \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 & -6 & -5 & -4 & -3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 8 & 6 & 4 & 4 \end{array} \right] & & M_{T \leftarrow T}(L) = \begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix} \end{aligned}$$

(d) Find the matrix representation of L with respect to T and S .

We need to form the matrix $[S \mid L(T)]$ and row reduce to get $[I \mid M_{S \leftarrow T}(L)]$. Since S is the standard basis, and we have already computed what L does to T , there is no work to do.

$$M_{S \leftarrow T}(L) = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 7 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}$$

(c) Find the matrix representation of L with respect to S and T .

We need to form the matrix $[T | L(S)]$ and row reduce to get $[I | M_{T \leftarrow S}(L)]$.

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 & 0 & 4 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & -1 & -1 & 0 & -1 & 3 & -2 & 4 \end{array} \right] \\ & \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 1 & -1 & 2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & -1 & 1 & -1 & 4 & -2 & 6 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -2 & -1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 2 & 6 \end{array} \right] \\ & \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 0 & -2 & -3 & -2 & -4 \\ 0 & 0 & 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 2 & 6 \end{array} \right] \quad M_{T \leftarrow S}(L) = \begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix} \end{aligned}$$

2. (p 290 #10) Let $L : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ be defined by $L(p(t)) = tp(t) + p(0)$.

(a) Find the matrix representation of L with respect to the bases $S = \{t, 1\}$ for \mathcal{P}_1 and $T = \{t^2, t, 1\}$ for \mathcal{P}_2 .

Comment: This should be easy enough to do directly, without forming the matrix $[T | L(S)]$.

$$L(at + b) = t(at + b) + b = at^2 + bt + b \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ b \end{bmatrix} \quad M_{T \leftarrow S}(L) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) Find the matrix representation of L with respect to the bases $S' = \{t + 1, t - 1\}$ for \mathcal{P}_1 and $T' = \{t^2 + 1, t - 1, t + 1\}$ for \mathcal{P}_2 .

We need to form the matrix $[T' | L(S')]$ and row reduce to get $[I | M_{T' \leftarrow S'}(L)]$.

$$L(t + 1) = t(t + 1) + 1 = t^2 + t + 1 \quad L(t - 1) = t(t - 1) - 1 = t^2 - t - 1$$

$$\begin{aligned} & \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & -2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 1 & -3 \end{array} \right] \\ & \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{array} \right] \quad M_{T' \leftarrow S'}(L) = \begin{bmatrix} 1 & 1 \\ 1/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix} \end{aligned}$$

To check this, we have

$$L(t + 1) = 1(t^2 + 1) + \frac{1}{2}(t - 1) + \frac{1}{2}(t + 1) = t^2 + t + 1 \quad L(t - 1) = 1(t^2 + 1) + \frac{1}{2}(t - 1) - \frac{3}{2}(t + 1) = t^2 - t - 1$$