1. (6.1 #8) (a) The characteristic polynomial of $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$ is $(\lambda-1)(\lambda+2)-4 = (\lambda-2)(\lambda+3)$. Eigenvectors corresponding to the eivenvalues $\lambda = 2$ and $\lambda = -3$ are (respectively) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) The characteristic polynomial of $\begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2 + 9$. Since this polynomial has no real roots, there are no eigenvalues (and therefore no eigenvectors).

(c) The characteristic polynomial of $A = \begin{bmatrix} 4 & 2 & -4 \\ 1 & 5 & -4 \\ 0 & 0 & 6 \end{bmatrix}$ is $(\lambda-6)[(\lambda-4)(\lambda-5)-2] = (\lambda-6)^2(\lambda-3)$. The eigenvector corresponding to the eivenvalue $\lambda = 3$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = 6$ we have $\lambda I - A = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and this has nullity 1. Then $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the only associated eigenvector. *(This means that the matrix cannot be diagonalized.)*

(d) The characteristic polynomial of $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is $\lambda^3 + \lambda = \lambda(\lambda^2+1)$. The only eigenvalue is $\lambda = 0$, and finding a basis for the nullspace of the matrix gives the corresponding eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

2. (6.2 #10) (b) The characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ is $(\lambda-1)^2(\lambda-3)$. Note: The matrix $\lambda I - A$ has block form $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, so its determinant is $|B| \cdot |D|$, and B is 1×1, while D is a lower-triangular 2×2 matrix. As a result, the determinant of $\lambda I - A$ is the product of the terms on its main diagonal.

For $\lambda = 1$ there are independent eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, and for $\lambda = 3$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector.

Since we have enough independent eigenvectors to match the eigenvalues, the matrix A can be diagonalized by $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, giving $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(d) The characteristic polynomial of $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$ is $(\lambda-1)(\lambda-2)$. Eigenvectors corresponding to $\lambda = 1$ and $\lambda = 2$ are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ for $P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$.

3. (6.4 #15) The characteristic polynomial of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ is $\lambda(\lambda-4)$. Eigenvectors corresponding to $\lambda = 0$ and $\lambda = 4$ are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note that these eigenvectors are orthogonal (as guaranteed by Theorem 6.7), so to find an orthonormal basis of eigenvectors we just need to divide each eigenvector by its length. Then $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$ for the orthogonal matrix $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

4. (6.4 #20) The characteristic polynomial of $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is $(\lambda+1)^3 - 12(\lambda+1) - 16 = (\lambda-3)(\lambda+3)^2$.

For $\lambda = 3$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector.

For $\lambda = -3$ we have $\lambda I - A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, giving the solution $x_1 = -x_2 - x_3$. Our standard method of choosing a basis for this nullspace is to first choose $x_2 = 1$, $x_3 = 0$, and then choose $x_2 = 0$, $x_3 = 1$. Because we need an orthonormal basis, we can use the Gram-Schmidt process to change the second of these two vectors to one that is orthogonal to the first. Alternatively, since we know the equation $x_1 = -x_2 - x_3$ that determines the subspace, we can just require the additional condition that the second solution must be orthogonal to the first vector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. This adds the equation $-x_1 + x_2 = 0$, and solving the two equations simultaneously gives us the equations $x_1 = -\frac{1}{2}x_3$, $x_2 = -\frac{1}{2}x_3$. The corresponding basis vector $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ completes our list of three mutually orthogonal basis vectors. The final step is to divide by their lengths to get an orthogonal matrix $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ with $P^T A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.