

Similarity of Matrix Representations (Kolman, Hill section 4.5)

Theorem 4.12 (page 299) Let V and W be finite dimensional vector spaces, and let L be a linear transformation from V to W . For any vector \mathbf{x} in V , we use $L(\mathbf{x})$ to denote the action of L on \mathbf{x} . In order to represent the action of L by a matrix, we must first choose bases S for V and T for W .

The coordinate vector of \mathbf{x} relative to S is $[\mathbf{x}]_S$.

The matrix representation of L relative to S and T is $M_{T \leftarrow S}(L)$. (*This is my notation.*)

The coordinate vector of $L(\mathbf{x})$ relative to T is $[L(\mathbf{x})]_T$.

This gives us the equation

$$[L(\mathbf{x})]_T = M_{T \leftarrow S}(L) \cdot [\mathbf{x}]_S .$$

If we choose different bases S' for V and T' for W , then we need to find the relationship between the two different matrices that are used to represent L relative to the different bases.

The coordinate vector of \mathbf{x} relative to S' is $[\mathbf{x}]_{S'}$.

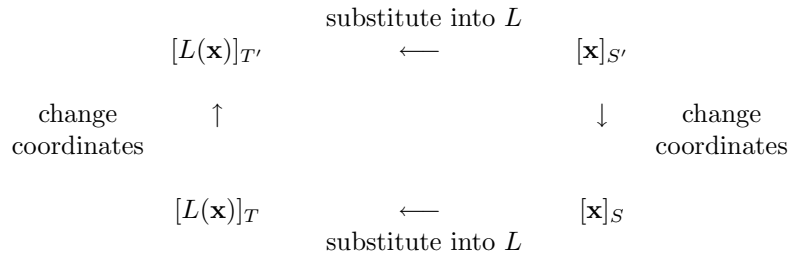
The matrix representation of L relative to S' and T' is $M_{T' \leftarrow S'}(L)$.

The coordinate vector of $L(\mathbf{x})$ relative to T' is $[L(\mathbf{x})]_{T'}$.

This gives us the equation

$$[L(\mathbf{x})]_{T'} = M_{T' \leftarrow S'}(L) \cdot [\mathbf{x}]_{S'} .$$

Given the coordinate vector $[\mathbf{x}]_{S'}$ of \mathbf{x} relative to S' , we can find $[L(\mathbf{x})]_{T'}$ in two different ways. We can either use the matrix representation $M_{T' \leftarrow S'}(L)$, or first change coordinates to $[\mathbf{x}]_S$, use the matrix representation $M_{T \leftarrow S}(L)$, and then change back from $[L(\mathbf{x})]_T$ to $[L(\mathbf{x})]_{T'}$. This is illustrated by the two different ways to go around the next diagram.



The matrix $P_{S \leftarrow S'}$ is the matrix for changing coordinates from S' to S , so $[\mathbf{x}]_S = P_{S \leftarrow S'} \cdot [\mathbf{x}]_{S'}$.

The matrix $P_{T' \leftarrow T}$ is the matrix for changing coordinates from T to T' , so $[L(\mathbf{x})]_{T'} = P_{T' \leftarrow T} \cdot [L(\mathbf{x})]_T$.

Using these equations to make the necessary substitutions gives us two equations.

$$[L(\mathbf{x})]_{T'} = M_{T' \leftarrow S'}(L) \cdot [\mathbf{x}]_{S'}$$

$$[L(\mathbf{x})]_{T'} = P_{T' \leftarrow T} \cdot [L(\mathbf{x})]_T = P_{T' \leftarrow T} \cdot M_{T \leftarrow S}(L) \cdot [\mathbf{x}]_S = P_{T' \leftarrow T} \cdot M_{T \leftarrow S}(L) \cdot P_{S \leftarrow S'} \cdot [\mathbf{x}]_{S'}$$

This shows that the corresponding matrices must be equal.

$$M_{T' \leftarrow S'}(L) = P_{T' \leftarrow T} \cdot M_{T \leftarrow S}(L) \cdot P_{S \leftarrow S'}$$

If we let $P_{S \leftarrow S'} = P$, $M_{T \leftarrow S}(L) = A$, and $P_{T \leftarrow T'} = Q$, then $P_{T' \leftarrow T} = Q^{-1}$, and so we get the final equation involving the two different matrix representations.

$$M_{T' \leftarrow S'}(L) = Q^{-1} A P$$

If $W = V$, $T = S$ and $T' = S'$, let $P = P_{S \leftarrow S'}$, $A = M_{S \leftarrow S}(L)$, and $B = M_{S' \leftarrow S'}(L)$. Then

$$B = P^{-1} A P .$$

Definition 4.6 (page 302) defines an $n \times n$ matrix B to be *similar* to an $n \times n$ matrix A if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Theorem 4.14 states that two matrices are similar if and only if they represent the same linear transformation (with respect to two ordered bases).

Example

Let $L : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be the linear transformation defined by $L(p(t)) = p''(t) - 4p'(t) + p(t)$.

(a) Find the matrix representations of L relative to the standard basis $S = \{t^3, t^2, t, 1\}$.

(b) Find the matrix representation of L relative to the basis $S' = \{t, t+1, t^2+t, t^3\}$.

(c) Find the relationship between the matrices in part (a) and part (b).

Solution: (a) To find the standard matrix, for each of the standard basis vectors we must compute its image under L and then find the coordinate vector of this image (relative to the standard basis).

$$\begin{aligned} L(t^3) &= 6t - 12t^2 + t^3 \\ L(t^2) &= 2 - 8t + t^2 \\ L(t) &= -4 + t \\ L(1) &= 1 \end{aligned}$$

This gives us the matrix representation relative to S .

$$M_{S \leftarrow S}(L) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -12 & 1 & 0 & 0 \\ 6 & -8 & 1 & 0 \\ 0 & 2 & -4 & 1 \end{bmatrix}$$

(b) To find the matrix relative to S' , for each of the basis vectors in S' we must compute its image under L and then find the coordinate vector of this image relative to S' .

$$\begin{aligned} L(t) &= -4 + t \\ L(t+1) &= -4 + (t+1) = t - 3 \\ L(t^2+t) &= 2 - 4(2t+1) + (t^2+t) = t^2 - 7t - 2 \\ L(t^3) &= 6t - 12t^2 + t^3 \end{aligned}$$

The next step is to find the coordinates of these vectors. The necessary equations lead to the following matrix, in which the first four columns are the standard coordinates of the basis vectors in S' , and the last four columns are the standard coordinates of $L(t)$, $L(t+1)$, $L(t^2+t)$, and $L(t^3)$. We need to row reduce the matrix.

$$\begin{aligned} &\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -12 \\ 1 & 1 & 1 & 0 & 1 & 1 & -7 & 6 \\ 0 & 1 & 0 & 0 & -4 & -3 & -2 & 0 \end{array} \right] \\ \rightsquigarrow &\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & -7 & 6 \\ 0 & 1 & 0 & 0 & -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ \rightsquigarrow &\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & 4 & -6 & 18 \\ 0 & 1 & 0 & 0 & -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

$$M_{S' \leftarrow S'}(L) = \begin{bmatrix} 5 & 4 & -6 & 18 \\ -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Since S is the standard basis, the matrix $P_{S \leftarrow S'}$ for the change of basis from S' to S simply consists of the coordinate vectors for the vectors in S' .

$$P_{S \leftarrow S'} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The matrix $P_{S' \leftarrow S}$ for the change of basis from S to S' is the inverse of $P_{S \leftarrow S'}$.

$$\begin{aligned} &\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ \rightsquigarrow &\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \\ \rightsquigarrow &\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

We now check the relationship

$$M_{S' \leftarrow S'}(L) = P_{S' \leftarrow S} \cdot M_{S \leftarrow S}(L) \cdot P_{S \leftarrow S'}$$

We have $M_{S \leftarrow S}(L) \cdot P_{S \leftarrow S'} =$

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 \\ -12 & 1 & 0 & 0 \\ 6 & -8 & 1 & 0 \\ 0 & 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -12 \\ 1 & 1 & -7 & 6 \\ -4 & -3 & -2 & 0 \end{bmatrix} \end{aligned}$$

and so $P_{S' \leftarrow S} \cdot (M_{S \leftarrow S}(L) \cdot P_{S \leftarrow S'}) =$

$$\begin{aligned} &\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -12 \\ 1 & 1 & -7 & 6 \\ -4 & -3 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 & -6 & 18 \\ -4 & -3 & -2 & 0 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M_{S' \leftarrow S'}(L). \end{aligned}$$

When we found the necessary coordinate vectors in part (b), we row reduced a matrix. In the method of part (c) we did the same calculation, but this time we used the inverse of the coefficient matrix in (b). The computations are really the same, and illustrate the fact that $M_{S' \leftarrow S'}(L) = P_{S' \leftarrow S} \cdot M_{S \leftarrow S}(L) \cdot P_{S \leftarrow S'}$.