MATH 520 Homework II 8/27/2004
Prof. John Beachy Due 9/13/2004

1. (1.3, p 41, #5) Let p, q be distinct prime numbers, and let $n = pq$. In $Z_n^\times$, let

$$H = \{[a] | a \equiv 1 \pmod{p}\} \quad \text{and} \quad K = \{[b] \mid b \equiv 1 \pmod{q}\}.$$

Show that $HK = Z_n^\times$.

2. (1.3, p 41, #8) This exercise concerns subgroups of $\mathbf{Z} \times \mathbf{Z}$.
(a) For each positive integer $n > 1$, let $C_n = \{(a,b) \in \mathbf{Z} \times \mathbf{Z} | a \equiv b \pmod{n}\}$. Show that C_n is a subgroup of $\mathbf{Z} \times \mathbf{Z}$.
(b) Let $D = \{(a,a) | a \in \mathbf{Z}\}$ be the "diagonal" subgroup of $\mathbf{Z} \times \mathbf{Z}$. Show that every subgroup of $\mathbf{Z} \times \mathbf{Z}$ that contains D has the form C_n, for some positive integer n.
(c) Show that $C_n \cong \mathbf{Z} \times \mathbf{Z}$.

3. (1.3, p 42, #10) Without writing down all 60 elements of A_5, describe the possible cycle structures and how many of each kind there are.

4. (1.3, p 42, #20) Given a permutation $\sigma \in S_n$, state a criterion on the cycle structure of σ that determines whether or not $\sigma = \tau^2$ for some $\tau \in S_n$. Use this criterion to find the values of n for which the alternating group A_n is precisely the set of squares in S_n.

5. (1.4, p 54, #5) Let G, G_1, and G_2 be groups.
(a) Show that if $\phi_1 : G \to G_1$ and $\phi_2 : G \to G_2$ are group homomorphisms, then so is $\phi : G \to G_1 \times G_2$ defined by $\phi(x) = (\phi_1(x), \phi_2(x))$, for all $x \in G$.
(b) Show that if $\phi : G \to G_1 \times G_2$ is any group homomorphism, then there exist group homomorphisms $\phi_1 : G \to G_1$ and $\phi_2 : G \to G_2$ such that ϕ has the form given in part (a).

6. (1.4, p 54, #13) Let N be a normal subgroup of the group G. Prove that G/N is abelian if and only if N contains all elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

7. (1.4, p 55, #17) Show that $\mathrm{SL}_2(\mathbf{Z}_3)$ is not isomorphic to S_4.

8. (1.4, p 54, #25) Let G be a group for which the only isomorphism from G into itself is the identity mapping. Prove that $x^2 = 1$ for all $x \in G$.

9. (1.4, p 54, #27) Let G be a group, and let N be a normal subgroup with $[G : N] = p$, where p is prime. Show that if H is a subgroup of G that is not contained in N, then $HN = G$ and $[H : H \cap N] = p$.

10. (1.4, p 54, #29) Let G be a group with subgroups H and K.
(a) Show that if $[G : H]$ and $[G : K]$ are finite, then so is $[G : H \cap K]$.
(b) Show that if $H \subseteq K$ and both $[G : K]$ and $[K : H]$ are finite, then we have $[G : H] = [G : K][K : H]$.