

This set of problems is worth 50 points. You may use the library (or internet) but may not discuss the problems with anyone (in the class or not).

1. (10 pts; Section 3.3 #7) Let K be a field, and let $F = K(u)$, where u is transcendental over K . If E is a field such that $K \subset E \subseteq F$, then show that u is algebraic over E .
2. (10 pts; Section 3.3 #14) Let F be an algebraic extension of K , and let S be a subset of F such that $S \supseteq K$, S is a vector space over K , and $s^n \in S$ for all $s \in S$ and all positive integers n . Prove that if $\text{char}(K) \neq 2$, then S is a subfield of F .
3. (10 pts; Section 3.4, #11, 12) Show that the splitting field of $x^4 - 2$ over \mathbf{Q} is $\mathbf{Q}(\sqrt[4]{2}, i)$. Use Exercise 3.4.8 to show that there are at most eight distinct automorphisms of the splitting field $\mathbf{Q}(\sqrt[4]{2}, i)$ of $x^4 - 2$ over \mathbf{Q} .
4. (10 pts; Section 3.5, #4) Show that $x^3 - x - 1$ and $x^3 - x + 1$ are irreducible over $GF(3)$. Construct their splitting fields and explicitly exhibit the isomorphism between these splitting fields.
5. (10 pts; Section 3.5, #9) Show that $x^p - x + a$ is irreducible over $GF(p)$ for all nonzero elements $a \in GF(p)$.