MATH 520 | Homework I | 8/29/2007
Prof. John Beachy | | Due 9/05/2007

1. (1.1, p22, #7) Let G be a set with an associative binary operation $*$. Prove that G is a group if (i) there exists a *left identity* $e \in G$ such that $e * a = a$ for each $a \in G$, and (ii) for each $a \in G$ there exists a *left inverse* $b \in G$ such that $b * a = e$.

2. (1.1, p22, #8) Prove that if G is a group and $a, b \in G$, then the equations $ax = b$ and $xa = b$ have unique solutions. Conversely, prove that if G is a nonempty set with an associative binary operation in which the equations $ax = b$ and $xa = b$ have solutions for all $a, b \in G$, then G is a group.

3. (1.1, p22, #9) Let S be a nonempty finite set with a binary operation $*$ that satisfies the associative law. Show that S is a group if $a * b = a * c$ implies $b = c$ and $a * c = b * c$ implies $a = b$ for all $a, b, c \in S$. What can you say if S is infinite?

4. (1.1, p23, #20) Let F be a field with q elements. Find $|\,\mathrm{GL}_n(F)|$.

5. (1.2, p32, #13) Show that in $\mathrm{SL}_2(\mathbf{Z}_3)$ the elements $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ generate a subgroup isomorphic to the quaternion group Q_8.

6. (1.2, p33, #18) Prove that $\sum_{d|n} \varphi(d) = n$ for any positive integer n.

7. (1.2, p33, #19) Let $n = 2^k$ for $k > 2$. Prove that $\mathbf{Z}_n^\times$ is not cyclic.
Hint: Show that ± 1 and $(n/2) \pm 1$ satisfy the equation $x^2 = 1$, and that this is impossible in any cyclic group.

8. (1.2, p33, #20) Let G be a group with p^k elements, where p is a prime number and k is a positive integer. Prove that G has a subgroup of order p.