

1. (1.3, p 43, #4) Let  $p, q$  be distinct prime numbers, and let  $n = pq$ . In  $\mathbf{Z}_n^\times$ , let  $H = \{[a] \mid a \equiv 1 \pmod{p}\}$  and  $K = \{[b] \mid b \equiv 1 \pmod{q}\}$ . Show that  $HK = \mathbf{Z}_n^\times$ .
2. (1.3, p 44, #6) Let  $a, b$  be positive integers, and let  $d = \gcd(a, b)$  and  $m = \text{lcm}[a, b]$ .
  - (a) Write  $d = sa + tb$ ,  $a = a'd$ , and  $b = b'd$  for  $s, t, a', b' \in \mathbf{Z}$ . Define  $\phi : \mathbf{Z}_a \times \mathbf{Z}_b \rightarrow \mathbf{Z}_d \times \mathbf{Z}_m$  by  $\phi([x]_a, [y]_b) = ([x - y]_d, [tb'x + sa'y]_m)$ . Prove that  $\phi$  is an isomorphism.
  - (b) Use Exercise 1.3.5 to prove that  $\mathbf{Z}_a \times \mathbf{Z}_b \cong \mathbf{Z}_d \times \mathbf{Z}_m$ .
3. (1.3, p 44, #9) Without writing down all 60 elements of  $A_5$ , describe the possible cycle structures and how many of each kind there are.
4. (1.3 p44, #14) Recall that center of  $G$  is  $Z(G) = \{g \in G \mid ga = ag \text{ for all } a \in G\}$ .
  - (a) Show that if  $n \geq 3$ , then the center of  $S_n$  is trivial.
  - (b) Find the center of  $D_n$ .
5. (1.3 p44, #15) Let  $G$  be a group with a normal subgroup  $N$ , and suppose that  $G$  is the semidirect product of  $N$  by the subgroup  $K$ . Show that if  $H$  is any subgroup with  $N \subseteq H \subseteq G$ , then  $H$  is the semidirect product of  $N$  by  $H \cap K$ .
6. (1.3 p44, #17) Show that in  $S_n$  the only elements which commute with the cycle  $(1, 2, \dots, n)$  are its powers.
7. (1.4, p 56, #5) Let  $G, G_1$ , and  $G_2$  be groups.
  - (a) Show that if  $\phi_1 : G \rightarrow G_1$  and  $\phi_2 : G \rightarrow G_2$  are group homomorphisms, then so is  $\phi : G \rightarrow G_1 \times G_2$  defined by  $\phi(x) = (\phi_1(x), \phi_2(x))$ , for all  $x \in G$ .
  - (b) Show that if  $\phi : G \rightarrow G_1 \times G_2$  is any group homomorphism, then there exist group homomorphisms  $\phi_1 : G \rightarrow G_1$  and  $\phi_2 : G \rightarrow G_2$  such that  $\phi$  has the form given in part (a).
8. (1.4, p 57, #13) Let  $N$  be a normal subgroup of the group  $G$ . Prove that  $G/N$  is abelian if and only if  $N$  contains all elements of the form  $aba^{-1}b^{-1}$  for  $a, b \in G$ .
9. (1.4 p57, #16) Show that  $D_4$  is isomorphic to the subgroup of all lower triangular matrices in  $\text{GL}_3(\mathbf{Z}_2)$ .
10. (1.4, p 57, #17) Show that  $\text{SL}_2(\mathbf{Z}_3)$  is not isomorphic to  $S_4$ .
11. (1.4, p 57, #20) Let  $F_{20}$  be the subgroup of  $\text{GL}_2(\mathbf{Z}_5)$  consisting of all matrices of the form  $\begin{bmatrix} 1 & 0 \\ x & a \end{bmatrix}$ , such that  $x, a \in \mathbf{Z}_5$  and  $a \neq 0$ . This group will be called the **Frobenius** group of degree 5.
  - (a) Show that the subset  $N$  of all matrices in  $F_{20}$  of the form  $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ , with  $c \in \mathbf{Z}_5$ , is a normal subgroup of  $G$ .
  - (b) Show that  $F_{20}/N$  is cyclic of order 4.
12. (1.4, p 58, #24) An infinite group is called **periodic** if each element has finite order.
  - (a) Show that  $\mathbf{Q}/\mathbf{Z}$  is a periodic group.
  - (b) Show that  $\mathbf{Q}/\mathbf{Z}$  contains elements of order  $n$ , for each positive integer  $n$ .
13. (1.4, p 58, #27) Let  $H$  and  $K$  be finite subgroups of the group  $G$ . Show that  $|HK| = \frac{|H||K|}{|H \cap K|}$ .
14. (1.4, p 58, #30) Let  $G$  be a group, with normal subgroups  $H$  and  $N$  such that  $HN = G$ . Prove that  $G/(H \cap N) \cong H/(H \cap N) \times (N/(H \cap N))$ .