

**§2.6 p102 #1:** Let  $G$  be a group and let  $N$  be a normal subgroup of  $G$ . For  $a, b \in G$ , let  $[a, b]$  denote the commutator  $aba^{-1}b^{-1}$ .

- (a) Show that  $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$ .
- (b) Show that  $N'$  is a normal subgroup of  $G$ .

**§2.6 p102 #5:** Find the derived subgroup  $D'_n$  of  $D_n$ . Show that  $D_n$  is a solvable group, for all  $n$ .

**§2.6 p102 #7:** Let  $G$  be the Frobenius group  $F_{20}$  of matrices of the form  $\begin{bmatrix} 1 & 0 \\ x & a \end{bmatrix}$  such that  $x \in \mathbf{Z}_5$  and  $a \in \mathbf{Z}_5^\times$ .

- (a) Find a composition series for  $G$ .
- (b) Find the descending series of commutator subgroups of  $G$ .

**§2.6 p102 #9:** Let  $p$  be a prime number. Prove that if  $G$  is a nonabelian group of order  $p^3$ , then  $G' = Z(G)$ .

**§2.6 p102 #10:** Let  $G$  be a group of order 780.

- (a) Prove that  $G$  is not simple.
- (b) Assume that  $G$  is not solvable. What are the composition factors of  $G$ ? You may assume that the only nonabelian simple group of order  $\leq 60$  is  $A_5$ .

**§2.6 p103 #12:** Let  $G$  be a group. A subgroup  $H$  of  $G$  is called a **characteristic** subgroup if  $\phi(H) \subseteq H$  for all  $\phi \in \text{Aut}(G)$ . (You could also use the term **fully invariant**.)

- (i) Prove that if  $H$  is a characteristic subgroup of  $G$ , the  $\phi(H) = H$  for all  $\phi \in \text{Aut}(G)$ .
- (a) Prove that any characteristic subgroup is normal.
- (b) Prove that the center of any group is a characteristic subgroup.
- (c) Prove that the commutator subgroup is always a characteristic subgroup.
- (d) Prove that any normal Sylow  $p$ -subgroup is a characteristic subgroup.
- (e) Prove that if  $H$  is a normal subgroup of  $G$ , and  $K$  is a characteristic subgroup of  $H$ , then  $K$  is normal in  $G$ .
- (ii) Prove that if  $K \subseteq H \subseteq G$ , where  $H$  is a characteristic subgroup of  $G$  and  $K$  is a characteristic subgroup of  $H$ , then  $K$  is a characteristic subgroup of  $G$ .
- (iii) Prove that the  $n$ th derived subgroup of  $G$  is a characteristic subgroup of  $G$ .