Math 521 FINAL EXAM 5/11/2005

J. Beachy Each question is worth 50 points.

1. State **3** of the following theorems.

(1) The Hilbert basis theorem

(2) The Artin-Wedderburn theorem

(3) Maschke's theorem

(4) The Krull-Schmidt thoerem

(5) The Chinese remainder theorem

(6) The fundamental structure theorem for finitely generated modules over a principal ideal domain

2. Solve Part A **or** Part B.

Part A. Let R be a ring. For elements of R,

(a) state the following definitions: *nilpotent element*, *idempotent element*, *regular element*, *zero divisor*.

(b) Which of the following sets of elements of R are closed under addition? (Give a proof or a counterexample.)

The set of all nilpotent elements; the set of all idempotent elements; the set of all regular elements; the set of all

zero divisors.

Part B. Let D be an integral domain with prime ideal P.

(a) Describe the construction of the localization D_P of D at P.

(b) Describe the connection between the ideals of D and those of D_P.

(c) Prove that if D is a Noetherian ring, then so is the localization D_P.

3. Solve Part A **or** Part B.

Part A. Let R be a ring, and let M be a left R-module.

(a) Complete this definition: M is a *simple* module if ...

(b) Prove that M is simple if and only if $M = Rm$, for all nonzero $m \in M$.

(c) Prove that M is simple if and only if $M \cong R/A$ for a maximal left ideal $A \subseteq R$.

(d) State and prove Schur's lemma.

Part B. Let R be a ring, and let M be a left R-module.

(a) Complete this definition: M is an *Artinian* module if ...

(b) Prove that if N is a submodule of M, then M is Artinian if and only if both N and M/N are Artinian.

(c) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.

4. Solve Part A **or** Part B.

Part A

(a) State the definition of the tensor product of two modules.

(b) Show that it follows from the definition that the tensor product is unique up to isomorphism.

(c) Show that $\mathbf{Z}_n \otimes_Z \mathbf{Z}_n$ is isomorphic to $\mathbf{Z}_n$. (Hint: you might show that $1 \otimes 1$ is a generator.)

Part B

(a) State the definitions of *projective* module and *injective* module.

(b) State Baer's criterion for injectivity.

(c) Prove that $\mathbf{Z}_n$ is injective as a left module over $\mathbf{Z}_n$. (That is, that $\mathbf{Z}_n$ is a self-injective ring.)

Extra credit (20 points):

Find the prime and maximal ideals of the ring of 2×2 lower triangular matrices over $\mathbf{Z}$. (Justify your answer.)

OR

Write out the solution to one of your favorite problems. (Points will be adjusted for difficulty.)