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Takehome

5.3.2. (15 pts) Let I, J be ideals of the commutative ring R. Show that if $\sqrt{I} + \sqrt{J} = R$, then $I + J = R$.

Solution: Let $a \in \sqrt{I}$ and $b \in \sqrt{J}$ with $a + b = 1$. There exist m, n with $a^m \in I$ and $b^n \in J$, and then $(a+b)^{m+n-1} = 1$. Expanding this expression we have a sum of terms of the form $a^i b^j$, with $i+j = m+n-1$ and the appropriate binomial coefficient. All of the first terms in which $i \geq m$ belong to I, and we denote their sum by x. The remaining terms, with $i \leq m-1$, and hence $j \geq n$, belong to J, and we denote their sum by y. Thus we have $x + y = 1$, for $x \in I$ and $y \in J$, and so $I + J = R$.

5.3.3. (15 pts) Prove that that if I, Q are ideals of the commutative ring R with $I \subseteq Q$, then Q is a primary ideal of R if and only if Q/I is a primary ideal of R/I.

Solution: The proof can be modeled on the corresponding one for prime ideals. In that case, Q is prime if and only if R/Q is an integral domain, and so we only need to observe that $(R/I)/(Q/I)$ is isomorphic to R/Q. Thus we need a criterion for being primary that can be applied to the factor ring R/Q.

The ideal Q is a primary ideal of R if it satisfies the following condition for all $a, c \in R$: $ac \in Q$ implies $a \in Q$ or $c^n \in Q$, for some $n \in \mathbf{Z}^+$. This translates into the following condition in R/Q. The ideal Q is a primary ideal of R if and only if in the factor ring R/Q we have the following condition for all elements $\overline{a}, \overline{c} \in R/Q$: $\overline{ac} = 0$ implies $\overline{a} = 0$ or $\overline{c}^n = 0$, for some $n \in \mathbf{Z}^+$. It then follows as above that Q is a primary ideal of R if and only if Q/I is a primary ideal of R/I.

5.3.5. (20 pts) Let F be a field, and consider the ideal $I = \left(x^2, xy\right)$ of $F[x, y]$.

(a) Show that I is not a primary ideal. (b) Show that $I = (x) \cap \left(x^2, y\right)$.

(c) Show that $\left(x^2, ax + y\right)$ is a primary ideal, for any $a \in F$, and show that $I = (x) \cap \left(x^2, ax + y\right)$, so that I can be represented in infinitely many ways as an intersection of primary ideals.

Solution: (a) The ideal I contains the product xy, but does not contain x and does not contain any power of y. Thus I not a primary ideal.

(b) and (c) Since $xy = x(ax + y) - ax^2$, we have $I \subseteq (x) \cap \left(x^2, ax + y\right)$ because the generators x^2 and xy belong to both ideals. On the other hand, if $f(x, y) \in (x^2, ax + y)$, then $f(x, y) = g(x, y)x^2 + h(x, y)(ax + y)$, or $f(x, y) = g(x, y)x^2 + ah(x, y)x + h(x, y)y$. If $f(x, y) \in (x)$, then x must be a factor of $h(x, y)$, showing that $f(x, y)$ has the form $f(x, y) = (g(x, y) + ak(x, y))x^2 + k(x, y)xy$, and thus $f(x, y) \in (x^2, xy)$. The special case $a = 0$ proves (b).

To prove that $(x^2, ax + y)$ is primary it is helpful to first prove the following result:

If I is an ideal of R such that $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R.

Proof: Let $a, b \in R$ with $ab \in I$. If $b^n \in I$ for some n, we are done. If not, then $b \notin \sqrt{I}$, and so b is invertible modulo $\sqrt{I}$ since $R/\sqrt{I}$ is a field. Since $\sqrt{I}/I$ is the Jacobson radical of R/I, it follows that b is invertible module I, and therefore $ab \in I$ implies $a \in I$.

We have shown that $xy \in (x^2, ax + y)$, and thus $y^2 = y(ax + y) - axy \in (x^2, ax + y)$. Therefore (x, y) is contained in the radical of $(x^2, ax + y)$, and so $(x, y) = \sqrt{(x^2, ax + y)}$ since (x, y) is maximal. It follows from the above result that $(x^2, ax + y)$ is primary.

An alternate proof can be given by making use of the automorphism of $F[x, y]$ that is defined by substituting $ax + y$ for y. This is a ring homomorphism, and substituting $y - ax$ defines an inverse. Under this automorphism, the ideal (x^2, y) corresponds to $(x^2, ax + y)$, and we already know that (x^2, y) is primary since modulo (y) it is a power of a maximal ideal.

In class

1. (25 pts) State **3** of the following definitions: (a) projective module; injective module; (b) the tensor product of modules M_R and ${}_RN$; (c) (for any ring) prime ideal; primitive ideal; maximal ideal; (d) the biendomorphism ring of the module ${}_RM$.

2. (25 pts) State **3** of the following theorems, including any necessary definitions not given in #1:
(a) the Hilbert basis theorem; (b) the Jordan–Hölder theorem; (c) the Krull–Schmidt theorem; (d) the fundamental theorem for finitely generated modules over a PID; (e) the Artin–Wedderburn theorem; (f) Hopkins's theorem; (g) the Jacobson density theorem.

3. (100 points) Prove **4** of the following:

(a) Prove that if ${}_RM$ is a module, and N is a submodule of M, then M is Noetherian if and only if both N and M/N are Noetherian. *Solution:* This is Proposition 2.4.5 (a).

(b) Let M be a left module, and let M^n denote the direct sum of n copies of M, with elements written as column vectors. (i) Show that M^n is a module over the matrix ring $S = \mathrm{M}_n(R)$. (ii) Show that there is a one-to-one correspondence between S-submodules of M^n and R-submodules of M.

Solution: (i) Multiplication of a matrix with entries from R and a column vector with entries from M is a well-defined operation. The associativity of matrix multiplication can be extended to this situation–I did not expect a complete proof. I just expected you to mention the necessary equations: $(AB)X = A(BX)$, $A(X+Y) = AX + AY$, $(A+B)X = AX + BX$, $IX = X$. (ii) It is clear that if N is a submodule of M, then N^n, the set of all columns vectors whose entries come from N, is a submodule of M^n. The converse is more interesting to prove, and requires more work. If K is any submodule of M^n, let N be the set of all elements of M that occur as the first entry of an element of K that has zero's in all of the remaining entries. It is easy to check that N is a submodule of M, since it is clearly closed under addition and scalar multiplication. Finally, if x is any element of K, it is possible to show that each entry of x belongs to N, since we can multiply x by a matrix unit which shifts the entry x_i to the first position and annihilates every other entry.

(c) If R is any ring, and I is an ideal of R, prove that $(R/I) \otimes_R (R/I)$ is isomorphic to R/I (as abelian groups).

Solution: The proof is a special case of Exercise 7 in Section 2.6, but a direct proof can be given more easily. Let $\pi : R \to R/I$ be the natural projection. Then $\pi \otimes 1 : R \otimes_R (R/I) \to (R/I) \otimes_R (R/I)$ is onto, and its kernel is generated by elements of the form $r \otimes 0$ and $a \otimes x$, where $a \in \ker(\pi)$. Since $\ker(\pi) = I$, we have $a \otimes x = 1 \otimes ax = 1 \otimes 0 = 0$. Thus $\pi \otimes 1$ is an isomorphism, and we can combine this with the known isomorphism from $R \otimes_R (R/I)$ onto R/I to prove that $(R/I) \otimes_R (R/I)$ is isomorphic to R/I.

(d) Let R be a ring, with Jacobson radical J. (i) Prove that J cannot contain a nonzero idempotent element of R. (ii) Prove that if R is left Noetherian, then J cannot contain a nonzero idempotent ideal of R

Solution: Note the correction in the statement of part (i): J contains 0, which is definitely idempotent.

(i) If x is idempotent, then $e = 1 - x$ is also idempotent, and if $x \in J$, then e is invertible. Thus $e^2 = e$ implies that $e = 1$, and hence $x = 0$, a contradiction.

(ii) If $I \subseteq J$ and $I^2 = I$, then $JI = I$, and so we can apply Nakayama's lemma, since by assumption I is finitely generated. It follows that $I = (0)$, a contradiction.

(e) Let ${}_RM$ be a module, and let $f \in \text{End}_R(M)$. (i) Show that if $\ker(f^2) = \ker(f)$, then $\ker(f) \cap \text{Im}(f) = (0)$. (ii) Show that if $\text{Im}(f^2) = \text{Im}(f)$, then $\ker(f) + \text{Im}(f) = M$.

Solution: (i) Let $m \in \ker(f) \cap \text{Im}(f)$. Then there exists $x \in M$ with $m = f(x)$, and so $f^2(x) = f(m) = 0$, showing that $x \in \ker(f)$. By assumption, $x \in \ker(f^2)$, and so $m = f(0) = 0$.

(ii) Given $m \in M$, by assumption there exists $x \in M$ with $f(m) = f^2(x)$. Let $m = (m - f(x)) + f(x)$. Then $f(x) \in \text{Im}(g)$, and $m - f(x) \in \ker(f)$ since $f(m - f(x)) = f(m) - f^2(x) = 0$.

(f) Prove that the ring R is a prime ring if and only if $\text{M}_n(R)$ is a prime ring.

Solution: My solution has two ingredients. First, R is a prime ring if and only if $AB = (0)$ implies $A = (0)$ or $B = (0)$, for all ideals of R. Secondly, the ideals of $\text{M}_n(R)$ are in one-to-one correspondence with those of R, where the ideal A of R corresponds to the ideal $\text{M}_n(A)$ consisting of all $n \times n$ matrices with entries in A.

Let A and B be ideals of R. Since the product AB consists of sums of products of the form ab, for $a \in A$ and $b \in B$, each entry of any matrix in the product $\text{M}_n(A) \cdot \text{M}_n(B)$ belongs to AB. Thus $AB = (0)$ if and only if $\text{M}_n(A) \cdot \text{M}_n(B) = (0)$. It follows immediately that R is prime if and only if $\text{M}_n(R)$ is prime.

(g) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.

Solution: In a left Artinian ring, the Jacobson radical is nilpotent, so it consists of divisors of zero. We conclude that the radical is zero, and so R is semisimple. If $e \neq 1$ is a nonzero idempotent, the e is a divisor of zero since $e(1 - e) = 0$. Thus R cannot be a nontrivial sum of minimal left ideals, so ${}_RR$ is a simple module, and hence R is a division ring since (0) is a maximal left ideal.