

Directions: Choose two questions from part A, one from part B, and one from part C. The questions are worth 25 points each. I suggest **1**, **3**, **5**, and **8**.

Part A (choose two)

about central idempotents and direct sums of rings:

1. (a) Let R be a ring, let e_1 be an idempotent ($e_1^2 = e_1$) element in the center of R , and let $e_2 = 1 - e_1$. Show that Re_1 and Re_2 are subrings of R , and that each element of R can be represented uniquely in the form $ae_1 + be_2$, for some $a, b \in R$.

(b) Let R be a ring isomorphic to the direct sum $S \oplus T$ of rings S and T . Prove that $M_n(R)$ is isomorphic to the direct sum of the rings $M_n(S) \oplus M_n(T)$.

Hint: Find central idempotent elements e_1 and e_2 of R such that $e_1 + e_2 = 1$, where there are obvious ring isomorphisms between Re_1 and the ring $M_n(S)$, and between Re_2 and the ring $M_n(T)$.

about group rings:

2. Let F be a field, and let G be a finite group.

(a) Give the definition of the group ring FG . That is, describe the elements of FG , and give the addition and multiplication in FG .

(b) For a ring R , give the definition of the opposite ring R^{op} .

(c) Show that FG is isomorphic to $(FG)^{op}$. You may assume the fact that any group homomorphism from G into the group of units of $(FG)^{op}$ will extend uniquely to a ring homomorphism from FG into $(FG)^{op}$.

about localization at a prime ideal:

3. Let D be an integral domain, and let P be a prime ideal of D .

(a) Describe the construction of the localization D_P of D at P . Describe the connection between the ideals of D and those of D_P .

(b) Prove that if every ideal of D is finitely generated, then the same holds for every ideal of the localization D_P .

about polynomial rings and Gauss's lemma:

4. (a) Show that if P is a prime ideal of the commutative ring R , then $P[x]$ is a prime ideal of the polynomial ring $R[x]$. (You may assume that $P[x]$, the set of all polynomials with coefficients in P , is an ideal of $R[x]$.)

(b) Use part (a) to prove Gauss's lemma. That is, prove that if D is a unique factorization domain, then the product of two primitive polynomials in $D[x]$ is again a primitive polynomial. (Recall that a nonconstant polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ in $D[x]$ is called primitive if there is no irreducible element $p \in D$ such that $p \mid a_i$ for all i .)

Part B (choose one)

about simple modules:

5. Let M be a left R -module.

- (a) Prove that M is a simple module if and only if $M = Rm$, for all nonzero $m \in M$.
- (b) Prove that M is simple if and only if $M \cong R/A$ for a maximal left ideal $A \subseteq R$.
- (c) (Schur's lemma) Prove that if M is simple, then $\text{End}_R(M)$ is a division ring.

about the homework problem on splitting, and a generalization:

6. Let M be a left R -module, and let f be an R -endomorphism of M .

- (a) Prove that if $f^2 = f$, then $M = \ker(f) \oplus \text{Im}(f)$.
- (b) Prove that if there exists an integer n such that $\text{Im}(f^n) = \text{Im}(f^{n+1}) = \dots = \text{Im}(f^{2n})$ and $\ker(f^n) = \ker(f^{n+1}) = \dots = \ker(f^{2n})$, then $M = \ker(f^n) \oplus \text{Im}(f^n)$.

about a new definition related to the socle:

7. Let M be a left R -module. A submodule $K \subseteq M$ is said to be *essential* if $K \cap N \neq (0)$ for all nonzero submodules $N \subseteq M$.

- (a) Show that \mathbf{Z} is an essential submodule of \mathbf{Q} .
- (b) Give the definition of the socle of M , and prove that it is equal to the intersection of all essential submodules of M .

Part C (no choice)

about the modular law, and an application:

8. Let M be a left R -module, with a submodule N .

- (a) Prove the modular law. That is, prove that if $K \subseteq L$ are any submodules of M , then $L \cap (K + N) = K + (L \cap N)$.
- (b) Prove that if $K \subseteq L$ are any submodules of M such that $K \cap N = L \cap N$ and $(K + N)/N = (L + N)/N$, then $K = L$.

The final exam is scheduled for Wednesday, May 5, at 10:00 am. The first part, worth 25%, consists of three takehome questions on primary decomposition in commutative rings. The solutions are due at the start of the exam. The remaining 75% will be in-class questions from Sections 2.4–2.7 and 3.1–3.3.

5.3.2. (15 pts) Let I, J be ideals of the commutative ring R . Show that if $\sqrt{I} + \sqrt{J} = R$, then $I + J = R$.

5.3.3. (15 pts) Prove that that if I, Q are ideals of the commutative ring R with $I \subseteq Q$, then Q is a primary ideal of R if and only if Q/I is a primary ideal of R/I .

5.3.5. (20 pts) Let F be a field, and consider the ideal $I = (x^2, xy)$ of $F[x, y]$.

(a) Show that I is not a primary ideal.

(b) Show that $I = (x) \cap (x^2, y)$.

(c) Show that $(x^2, ax + y)$ is a primary ideal, for any $a \in F$, and show that $I = (x) \cap (x^2, ax + y)$, so that I can be represented in infinitely many ways as an intersection of primary ideals.

1. (25 pts) State **3** of the following definitions:
 - (a) projective module; injective module;
 - (b) the tensor product of modules M_R and ${}_R N$;
 - (c) (for any ring) prime ideal; primitive ideal; maximal ideal;
 - (d) the biendomorphism ring of the module ${}_R M$.

2. (25 pts) State **3** of the following theorems, including any necessary definitions not given in question #1:
 - (a) the Hilbert basis theorem;
 - (b) the Jordan–Hölder theorem;
 - (c) the Krull–Schmidt theorem;
 - (d) the fundamental theorem for finitely generated modules over a PID;
 - (e) the Artin–Wedderburn theorem;
 - (f) Hopkins’s theorem;
 - (g) the Jacobson density theorem.

3. (100 points) Prove **4** of the following:
 - (a) Prove that if ${}_R M$ is a module, and N is a submodule of M , then M is Noetherian if and only if both N and M/N are Noetherian.
 - (b) Let M be a left module, and let M^n denote the direct sum of n copies of M , with elements written as column vectors.
 - (i) Show that M^n is a module over the matrix ring $S = M_n(R)$.
 - (ii) Show that there is a one-to-one correspondence between S -submodules of M^n and R -submodules of M .
 - (c) If R is any ring, and I is an ideal of R , prove that $(R/I) \otimes_R (R/I)$ is isomorphic to R/I (as abelian groups).
 - (d) Let R be a ring, with Jacobson radical J .
 - (i) Prove that J cannot contain a nonzero idempotent element of R .
 - (ii) Prove that if R is left Noetherian, then J cannot contain a nonzero idempotent ideal of R (i.e. $I^2 = I$ implies $I = (0)$, for any ideal $I \subseteq J$).
 - (e) Let ${}_R M$ be a module, and let $f \in \text{End}_R(M)$.
 - (i) Show that if $\ker(f^2) = \ker(f)$, then $\ker(f) \cap \text{Im}(f) = (0)$.
 - (ii) Show that if $\text{Im}(f^2) = \text{Im}(f)$, then $\ker(f) + \text{Im}(f) = M$.
 - (f) Prove that the ring R is a prime ring if and only if $M_n(R)$ is a prime ring.
 - (g) Prove that a left Artinian ring with no nonzero divisors of zero is a division ring.