

1. Find the prime ideals of the ring $R = \begin{bmatrix} \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$.
2. Let $M = {}_R M$ be a left R -module, and let $U = {}_R U_S$ be an R - S -bimodule.
 - (a) Show that $\text{Hom}_R(U, M)$ has a natural structure as a left S -module, defined by $(sf)(u) = f(us)$, for all $s \in S$ and all $f \in \text{Hom}_R(U, M)$.
 - (b) Using this structure on $\text{Hom}_R(R, M)$, prove that $\text{Hom}_R(R, M) \cong M$ (as left R -modules).
3. Let K be a field, and let $R = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}$.
 - (a) Show that the following are simple left R -modules.
 $S_1 = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix} / \begin{bmatrix} 0 & 0 \\ K & K \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$, $S_3 = \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}$
 - (b) Show that $S_1 \not\cong S_2$, and $S_2 \cong S_3$.
4. Let S be a commutative ring, and let M be an S -module. We define the *trivial extension* of M by S to be the set of all formal matrices

$$R = \left\{ \begin{bmatrix} s & 0 \\ m & s \end{bmatrix} \mid s \in S \text{ and } m \in M \right\} .$$

The notation $R = S \rtimes M$ is used, and this ring is sometimes called the *idealization* of M .

- (a) Show that R is a commutative ring.
- (b) If N is an S -submodule of M , and I is an ideal of S such that $IM \subseteq N$, we define

$$I \rtimes N = \left\{ \begin{bmatrix} a & 0 \\ x & a \end{bmatrix} \mid a \in I \text{ and } x \in N \right\} .$$

Show that $I \rtimes N$ is an ideal of R .

- (c) Show that the correspondence $N \leftrightarrow (0) \rtimes N$ determines a one-to-one correspondence between the set of S -submodules N of M and the set of ideals of R that are contained in $(0) \rtimes M$.