PRIME $M$-IDEALS, $M$-PRIME SUBMODULES, $M$-PRIME RADICAL AND $M$-BAER’S LOWER NILRADICAL OF MODULES

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Abstract. Let $M$ be a fixed left $R$-module. For a left $R$-module $X$, we introduce the notion of $M$-prime (resp. $M$-semiprime) submodule of $X$ such that in the case $M = R$, which coincides with prime (resp. semiprime) submodule of $X$. Other concepts encountered in the general theory are $M$-$m$-system sets, $M$-$n$-system sets, $M$-prime radical and $M$-Baer’s lower nilradical of modules. Relationships between these concepts and basic properties are established. In particular, we identify certain submodules of $M$, called “prime $M$-ideals”, that play a role analogous to that of prime (two-sided) ideals in the ring $R$. Using this definition, we show that if $M$ satisfies condition $H$ (defined latter) and $\text{Hom}_R(M, X) \neq 0$ for all modules $X$ in the category $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable $M$-injective modules in $\sigma[M]$ and prime $M$-ideals of $M$. Also, we investigate the prime $M$-ideals, $M$-prime submodules and $M$-prime radical of Artinian modules.

1. Introduction

All rings in this paper are associative with identity and modules are unitary left modules. Let $R$ be a ring and $X$ be an $R$-module. If $Y$ is a submodule (resp. proper submodule) of $X$ we write $Y \leq X$ (resp. $Y \subsetneq X$).

In the literature, there are many different generalizations of the notion of prime two-sided ideals to left ideals and also to modules. For instance, a proper left ideal $L$ of a ring $R$ is called prime if, for any elements $a$ and $b$ in $R$ such that $aRb \subseteq L$, either $a \in L$ or $b \in L$. Prime left ideals have properties reminiscent of prime ideals in commutative rings. For example, Michler [19] and Koh [12] proved that the ring $R$ is left Noetherian if and only if every prime left ideal is finitely generated. Moreover, Smith [20], showed...
that if $R$ is left Noetherian (or even if $R$ has finite left Krull dimension) then a left $R$-module $X$ is injective if and only if, for every essential prime left ideal $L$ of $R$ and homomorphism $\varphi : L \to X$, there exists a homomorphism $\theta : R \to X$ such that $\theta|_L = \varphi$. Let us mention another generalization of the notion of prime ideals to modules. Let $X$ be a left $R$-module. If $X \neq 0$ and $\text{Ann}_R(X) = \text{Ann}_R(Y)$ for all nonzero submodules $Y$ of $X$ then $X$ is called a prime module. A proper submodule $P$ of $X$ is called a prime submodule if $X/P$ is a prime module, i.e., for every ideal $I \subseteq R$ and every submodule $Y \subseteq X$, if $IY \subseteq P$, then either $Y \subseteq P$ or $IX \subseteq P$. The notion of prime submodule was first introduced and systematically studied by Dauns [7] and recently has received some attention. Several authors have extended the theory of prime ideals of $R$ to prime submodules (see [2, 3, 4, 7, 10, 15, 17, 18]). For example, the classical result of Cohens is extended to prime submodules over commutative rings, namely a finitely generated module is Noetherian if and only if every prime submodule is finitely generated (see [15, Theorem 8] and [11]) and also any Noetherian module contains only finitely many minimal prime submodules (see [18, Theorem 4.2]).

We assume throughout the paper $R_M$ is a fixed left $R$-module. The category $\sigma[M]$ is defined to be the full subcategory of $R$-Mod that contains all modules $R_X$ such that $X$ is isomorphic to a submodule of an $M$-generated module (see [21] for more detail).

Let $C$ be a class of modules in $R$-Mod, and let $\Omega$ be the set of kernels of $R$-homomorphisms from $M$ in to $C$. That is,

$$\Omega = \{K \subseteq M \mid \exists W \in C \text{ and } f \in \text{Hom}_R(M,W) \text{ with } K = \ker(f)\}.$$ 

Then the annihilator of $C$ in $M$, denoted by $\text{Ann}_M(C)$, is defined to be the intersection of all elements of $\Omega$, i.e., $\text{Ann}_M(C) = \bigcap_{K \in \Omega} K$.

Let $N$ be a submodule of $M$. Following Beachy [1], for each module $R_X$ we define

$$N \cdot X = \text{Ann}_X(C),$$

where $C$ is the class of modules $R_W$ such that $f(N) = (0)$ for all $f \in \text{Hom}_R(M, W)$. It follows immediately from the definition that

$$N \cdot X = (0) \text{ if and only if } f(N) = (0) \text{ for all } f \in \text{Hom}_R(M, X).$$

Clearly the class $C$ in definition of $N \cdot X$ is closed under formation of submodules and direct products, and so $N \cdot X$ is the smallest submodule $Y \subseteq X$ such that $N \cdot (X/Y) = (0)$.

The submodule $N$ of $M$ is called an $M$-ideal if there is a class $C$ of modules in $\sigma[M]$ such that $N = \text{Ann}_M(C)$. Note that although the definition of an $M$-ideal is given relative to the subcategory $\sigma[M]$, it is easy to check that $N$ is an $M$-ideal if and only if $N = \text{Ann}_M(C)$ for some class $C$ in $R$-Mod (see [1, Page 4651]).

In this article for a left $R$-module $X$, we introduce the notions of $M$-prime submodule, $M$-semi-prime submodule of $X$ and prime $M$-ideal of $M$ as follows:
Definition 1.1. Let $X$ be an $R$-module. A proper submodule $P$ of $X$ is called an $M$-prime submodule if for all submodules $N \leq M$, $Y \leq X$, if $N \cdot Y \subseteq P$, then either $N \cdot X \subseteq P$ or $Y \subseteq P$. An $R$-module $X$ is called an $M$-prime module if $(0) \nsubseteq X$ is an $M$-prime submodule. Also, a proper submodule $P$ of $X$ is called an $M$-semiprime submodule if for all submodules $N \leq M$, $Y \leq X$, if $N^{2} \cdot Y \subseteq P$, then $N \cdot Y \subseteq P$, where $N^{2} := N \cdot N$. An $R$-module $X$ is called an $M$-semiprime module if $(0) \nsubseteq X$ is an $M$-semiprime submodule.

Definition 1.2. A proper $M$-ideal $P$ of $M$ is called a prime $M$-ideal (resp. semiprime $M$-ideal) if there exists an $M$-prime module (resp. $M$-semiprime module) $\mu X$ such that $P = \text{Ann}_M(X)$.

It is clear that in case $M = R$, the notion of an $R$-prime submodule (resp. $R$-semiprime submodule) reduces to the familiar definition of a prime submodule (resp. semiprime submodule). Also, the notion of an $R$-ideal (resp. prime $R$-ideal) of $R$ reduces to the familiar definition of an ideal (resp. a prime ideal) of $R$.

Recently, the idea of $M$-prime module was introduced and extensively studied by Beachy [1] by defining a module $\mu X$ to be $M$-prime if $\text{Hom}_R(M, X) \neq 0$, and $\text{Ann}_M(Y) = \text{Ann}_M(X)$ for all submodules $Y \leq X$ such that $\text{Hom}_R(M, Y) \neq 0$. Also, he defined an $M$-ideal $P$ to be prime $M$-ideal if there exists an $M$-prime module $\mu X$ such that $P = \text{Ann}_M(X)$. Clearly, our definition of $M$-prime module is slightly different than Beachy, and hence, for the sake of clarity, for the remainder of the paper we will use the term “Beachy-$M$-prime module” (resp. “Beachy-$M$-ideal”) rather than “$M$-prime module” (resp. “prime $M$-ideal”) of Beachy [1], respectively.

In ring theory, prime ideals are closely tied to $m$-system sets (a nonempty set $S \subseteq R$ is said to be an $m$-system set if for each pair $a, b$ in $S$, there exists $r \in R$ such that $arb \in S$). The complement of a prime ideal is an $m$-system, and given an $m$-system set $S$, an ideal disjoint from $S$ and maximal with respect to this property is always a prime ideal. Moreover, for an ideal $I$ in a ring $R$, the set $\sqrt{I} := \{s \in R \mid \text{ every } m\text{-system containing } s \text{ meets } I\}$ equals the intersection of all the prime ideals containing $I$. In particular, $\sqrt{I}$ is a semiprime ideal in $R$ and $\sqrt{(0)}$ is called Baer-McCoy radical (or prime radical) of $R$ (see for example [14, Chapter 4], for more details). In this paper, we extend these facts for $M$-prime submodules. Relationships between these concepts and basic properties are established. In Section 2, among other results, for an $R$-module $X$ we define $M$-Baer-McCoy radical (or $M$-prime radical) of $X$, denoted $\text{rad}_M(X) = \sqrt{(0)}$, to be the intersection of all the $M$-prime submodules in $X$. Also, in Section 3, we extend the notion of nilpotent and strongly nilpotent element of modules to $M$-nilpotent and strongly $M$-nilpotent element of modules $X \in \sigma[M]$ for a fix module $M$. Also, for an $R$-module $X \in \sigma[M]$, we define $M$-Baer’s lower nilradical of $X$, denoted by $M$-Nil$_*(\mu X)$, to be the set of all strongly $M$-nilpotent elements of $X$. In particular, it is shown that if $M$ is projective in
σ[M], then for each \( X \in \sigma[M] \), \( \text{Nil}_* (M) \cdot X \subseteq M - \text{Nil}_* (M) X \subseteq \text{rad}_M (X) \) (see Proposition 3.6).

In Section 4, we rely on the prime \( M \)-ideals of \( M \) that play a role analogous to that of prime ideals in the ring \( R \). The module \( _RX \) is called \( M \)-injective if each \( R \)-homomorphism \( f : K \to X \) defined on a submodule \( K \) of \( M \) can be extended to an \( R \)-homomorphism \( \hat{f} : M \to X \) with \( f = \hat{f} i \), where \( i : K \to M \) is the natural inclusion mapping. We note that Baers criterion for injectivity shows that any \( R \)-injective module is injective in the category \( R \text{-Mod} \) of all left \( R \)-modules. It is well-known that if \( R \) is a commutative Noetherian ring, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective \( R \)-modules and prime ideals of \( R \). Gabriel showed in [8] that this one-to-one correspondence remains valid for any left Noetherian ring that satisfies what he called condition \( H \). In current terminology, a module \( _RX \) is said to be finitely annihilated if there is a finite subset \( x_1, \ldots, x_n \) of \( X \) with \( \text{Ann}_R (X) = \text{Ann}_R (x_1, \ldots, x_n) \). Then by definition the ring \( R \) satisfies condition \( H \) if and only if every cyclic left \( R \)-module is finitely annihilated. It follows immediately that, the ring \( R \) satisfies condition \( H \) if and only if every finitely generated left \( R \)-module is finitely annihilated. We note the stronger result due to Krause [13] that if \( R \) is left Noetherian, then there is a one-to-one correspondence between isomorphism classes of indecomposable injective \( R \)-modules and prime ideals of \( R \). In Section 4, by using our definition of prime \( M \)-ideal, we show that also there is Gabriels correspondence between indecomposable \( M \)-injective modules in \( \sigma[M] \) and our prime \( M \)-ideals.

Finally, in Section 5, we study the prime \( M \)-ideal, \( M \)-prime submodules and \( M \)-prime radical of Artinian modules. The prime radical of the module \( M \), denoted by \( P(M) \), is defined to be the intersection of all prime \( M \)-ideals of \( M \). Recall that a proper submodule \( P \) of \( M \) is virtually maximal if the factor module \( M/P \) is a direct sum of isomorphic simple modules. It is shown that if \( M \) is an Artinian \( M \)-prime module, then \( M \) is a homogeneous semisimple module (see Proposition 5.1). In particular, if \( M \) is an Artinian \( R \)-module such that it is projective in \( \sigma[M] \), then every prime \( M \)-ideal of \( M \) is virtually maximal and \( M/P(M) \) is a Noetherian \( R \)-module (see Theorem 5.6). Moreover, either \( P(M) = M \) or there exist primitive (prime) \( M \)-ideals \( P_1, \ldots, P_n \) of \( M \) such that \( P(M) = \bigcap_{i=1}^n P_i \) (see Theorem 5.7).

2. \( M \)-prime submodules and \( M \)-prime radical of modules

We begin this section with the following three useful lemmas.

Lemma 2.1 ([1, Proposition 1.6]). Let \( N \) be a submodule of \( M \). Then for any \( R \)-module \( X \), \( N \cdot X = (0) \) if and only if \( N \subseteq \text{Ann}_M (X) \).
Lemma 2.2 ([1, Proposition 1.9]). Let $N$ and $K$ be submodules of $M$.

(a) If $N \subseteq K$, then $N \cdot X \subseteq K \cdot X$ for all submodules $R X$.
(b) If $K$ is an $M$-ideal, then so is $N \cdot K$.
(c) The submodule $N \cdot M$ is the smallest $M$-ideal that contains $N$.
(d) If $N$ is an $M$-ideal, then $N \cdot K \subseteq N \cap K$.

Lemma 2.3. Let $Y_1, Y_2$ be submodules of $R X$. If $Y_1 \subseteq Y_2$, then $N \cdot Y_1 \subseteq N \cdot Y_2$, for each submodule $N$ of $M$.

Proof. Suppose $N \subseteq M$ and $Y_1, Y_2$ be submodules of $R X$ with $Y_1 \subseteq Y_2$. Then $N \cdot Y_1 = \text{Ann}_{Y_1}(C)$ and $N \cdot Y_2 = \text{Ann}_{Y_2}(C)$, where $C$ is the class of modules $R W$ such that $f(N) = (0)$ for all $f \in \text{Hom}_R(M, W)$. On the other hand $N \cdot Y_i = \bigcap_{K \in \Omega} K$ ($i = 1, 2$), where

$$\Omega_i = \{K \subseteq Y_i \mid \exists W \in C \text{ and } f \in \text{Hom}_R(Y_i, W) \text{ with } K = \ker(f)\}$$

Clearly, for each $f \in \text{Hom}_R(Y_2, W)$, $f|_{Y_1} \in \text{Hom}_R(Y_1, W)$, where $f|_{Y_1}$ is the restriction of $f$ on $Y_1$. Since $\ker(f|_{Y_1}) \subseteq \ker(f)$, we conclude that for each $K \in \Omega_2$, there exists $K' \in \Omega_1$ such that $K' \subseteq K$. Thus $N \cdot Y_1 \subseteq N \cdot Y_2$. □

The following evident proposition offers several characterizations of an $M$-prime module.

Proposition 2.4. Let $X$ be a nonzero $R$-module. Then the following statements are equivalent.

1. $X$ is an $M$-prime module.
2. For every submodule $N \subseteq M$ and every nonzero submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.
3. For every $M$-ideal $N \subseteq M$ and every nonzero submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.
4. For every nonzero submodules $Y_1, Y_2 \subseteq X$, Ann$_M(Y_1) = Ann_M(Y_2)$.
5. Every nonzero submodule $Y \subseteq X$ is an $M$-prime module.
6. For every nonzero submodule $Y \subseteq X$, $P = Ann_M(Y)$ is a prime $M$-ideal of $M$ and $P = Ann_M(X)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let $Y_1, Y_2$ be two nonzero submodules of $X$ and let $N_1 := Ann_M(Y_1)$, $N_2 := Ann_M(Y_2)$. Thus by Lemma 2.1, $N_1 \cdot Y_1 = (0)$ and $N_2 \cdot Y_2 = (0)$. Since $N_1, N_2$ are $M$-ideals, $N_1 \cdot X = N_2 \cdot X = (0)$ by (3). Thus $N_1 \subseteq Ann_M(X)$ and $N_2 \subseteq Ann_M(X)$. On the other hand $Ann_M(X) \subseteq N_1$ and $Ann_M(X) \subseteq N_2$. Thus $N_1 = N_2 = Ann_M(X)$.

(4) $\Rightarrow$ (5). Let $Y$ be a nonzero submodule of $X$. Assume that $N$ is a submodule of $M$ and $Z$ be a nonzero submodule of $Y$ such that $N \cdot Z = (0)$. So $N \subseteq Ann_M(Z)$. By (4), $Ann_M(Z) = Ann_M(X)$ and so it follows that $N \subseteq Ann_M(X)$ and hence $N \cdot X = (0)$. Since $N \cdot Y \subseteq N \cdot X$, so $N \cdot Y = (0)$. Thus $Y$ is an $M$-prime module.

(5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6) $\Rightarrow$ (4) are clear. □
By Proposition 2.4 and Lemma 2.6, it is clear.

**Proof.**

Example 2.8.

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Proof. (i). Let

(ii). For an

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Lemma 2.6 ([1, Proposition 2.2]). Let $X$ be an $R$-module such that $\text{Hom}_R(M, X) \neq 0$. Then the following statements are equivalent.

1. $X$ is a Beachy-$M$-prime module.
2. For every $M$-ideal $N$ of $M$ and every nonzero submodule $Y$ of $X$ with $M \cdot Y \neq (0)$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.
3. For each $m \in M \setminus \text{Ann}_M(X)$ and each $0 \neq f \in \text{Hom}_R(M, X)$, there exists $g \in \text{Hom}_R(M, f(M))$ such that $g(m) \neq 0$.
4. For any $M$-ideal $N \subseteq M$ and any $M$-generated submodule $Y \subseteq X$, if $N \cdot Y = (0)$, then $N \cdot X = (0)$.

**Proposition 2.7.** Let $X$ be an $R$-module such that $\text{Hom}_R(M, X) \neq 0$. If $X$ is an $M$-prime module then $X$ is a Beachy-$M$-prime module.

**Proof.** By Proposition 2.4 and Lemma 2.6, it is clear. \qed

The following example shows that the converse of Proposition 2.7 is not true in general.

**Example 2.8.** Let $R = \mathbb{Z}$. For each prime number $p$, $\text{Hom}_\mathbb{Z}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \neq 0$ and for each proper $\mathbb{Z}$-submodule $Y \nsubseteq \mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty} \cdot Y = (0)$, since $\text{Hom}_\mathbb{Z}(\mathbb{Z}_{p^\infty}, Y) = (0)$. Thus by Lemma 2.6, $\mathbb{Z}_{p^\infty}$ is a Beachy-$\mathbb{Z}_{p^\infty}$-prime module but it is not an $\mathbb{Z}_{p^\infty}$-prime module, since $\mathbb{Z}_{p^\infty} \cdot \mathbb{Z}_{p^\infty} \neq (0)$.

**Lemma 2.9** ([1, Proposition 5.5]). Assume that $M$ is projective in $\sigma[M]$, and let $N$ be any submodule of $M$. The following conditions hold for any module $R \cdot X$ in $\sigma[M]$ and any submodule $Y \subseteq X$.

a. $N \cdot X = \sum_{f \in \text{Hom}_R(M, X)} f(N)$.

b. $N \cdot (X/Y) = (0)$ if and only if $N \cdot X \subseteq Y$.

c. If $N = \text{Ann}_M(X/Y)$, then $\text{Ann}_M(X/(N \cdot X)) = N$.

**Proposition 2.10.** Assume that $M$ is projective in $\sigma[M]$, and let $R \cdot X \in \sigma[M]$. Then

(i) For a submodule $P \nsubseteq X$, if $P$ is an $M$-prime submodule of $X$, then $X/P$ is an $M$-prime module.

(ii) For an $M$-ideal $P \nsubseteq M$, the following conditions are equivalent.

1. $P$ is a prime $M$-ideal.

2. $P$ is an $M$-prime submodule of $M$.

3. $M/P$ is an $M$-prime module.

**Proof.** (i). Let $N$ be a submodule of $M$ and $Y/P$ be a nonzero submodule of $X/P$ such that $N \cdot (Y/P) = (0)$. By Lemma 2.9(b), $N \cdot Y \subseteq P$. Since $P$ is an
M-prime submodule, either $N \cdot X \subseteq P$ or $Y \subseteq P$. If $Y \subseteq P$, then $Y/P = (0)$, a contradiction. Thus $N \cdot X \subseteq P$ and so $N \cdot (X/P) = (0)$ by Lemma 2.9(b).

Thus by Proposition 2.4, $X/P$ is an $M$-prime module.

(ii) $(1) \Rightarrow (2)$. Suppose that $P$ is a prime $M$-ideal and $N \cdot K \subseteq P$, for an $M$-ideal $N$ and submodule $K$ of $M$ with $K \not\subseteq P$. By assumption there is an $M$-prime module $X$ with $P = \text{Ann}_M(X)$, and so there exists $f \in \text{Hom}_R(M/P, X)$ with $f((K+P)/P) \neq (0)$. Since $N \cdot K \subseteq P$, we have $N \cdot K \subseteq P \cap K$. Now Lemma 2.9(b) implies that $N \cdot (K/(P \cap K)) = (0)$ and hence $N \cdot f((K + P)/P) = (0)$ (since $(K + P)/P \cong K/(P \cap K)$). Since $X$ is an $M$-prime module, $N \cdot X = (0)$ by Proposition 2.4, and so $N \subseteq P$ (since $P = \text{Ann}_M(X)$).

$(2) \Rightarrow (3)$. Let $N$ be an $M$-ideal and $K/P$ be a nonzero submodule of $M/P$ such that $N \cdot (K/P) = (0)$. Since $M$ is projective in $\sigma[M]$, so $N \cdot K \subseteq P$ by Lemma 2.9(b). Now by (2) either $N \subseteq P$ or $K \subseteq P$. Since $K/P \neq (0)$, so $K \not\subseteq P$ and hence $N \subseteq P$. On the other hand $N \cdot M = N$, since $N$ is an $M$-ideal. Thus $N \cdot M \subseteq P$ and hence by Lemma 2.9(b), $N \cdot (M/P) = (0)$. Now $M/P$ is an $M$-prime module by Proposition 2.4.

$(3) \Rightarrow (1)$. Since $P$ is an $M$-ideal, $P = \text{Ann}_M(M/P)$ and since $M/P$ is an $M$-prime module, we conclude that $P$ is a prime $M$-ideal.

The following example shows that even in the case the $R$-module $M$ is projective in $\sigma[M]$, an $M$-prime module need not be a Beachy-$M$-prime module.

**Example 2.11.** Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$ as $\mathbb{Z}$-module. Then it is easy to check that $\mathbb{Q}$ is projective in $\sigma[\mathbb{Q}]$. Clearly, for each prime number $p$, $\mathbb{Z}_p$ is a $\mathbb{Q}$-prime module, but it is not a Beachy-$\mathbb{Q}$-prime module, since $\text{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z}_p) = (0)$.

Now we have to adapt the notion of an $M$-m-system set to modules $R X$ (Behboodi in [2], has generalized the notion of $m$-system of rings to modules).

**Definition 2.12.** Let $X$ be an $R$-module. A nonempty set $S \subseteq X \setminus \{0\}$ is called an $M$-m-system if, for each submodule $N \subseteq M$, and for all submodules $Y, Z \subseteq X$, if $(Y + Z) \cap S \neq \emptyset$ and $(Y + N \cdot Z) \cap S \neq \emptyset$, then $(Y + N \cdot Z) \cap S \neq \emptyset$.

**Corollary 2.13.** Let $X$ be an $R$-module. Then a submodule $P \subseteq X$ is $M$-prime if and only if $X \setminus P$ is an $M$-m-system.

**Proof.** $(\Rightarrow)$. Suppose $S = X \setminus P$. Let $N$ be a submodule of $M$ and $Y, Z$ be submodules of $X$ such that $(Y + Z) \cap S \neq \emptyset$ and $(Y + N \cdot Z) \cap S \neq \emptyset$. If $(Y + N \cdot Z) \cap S = \emptyset$ then $Y + N \cdot Z \subseteq P$. Hence $N \cdot Z \subseteq P$ and since $P$ is an $M$-prime submodule, $Z \subseteq P$ or $N \cdot X \subseteq P$. It follows that $(Y + Z) \cap S = \emptyset$ or $(Y + N \cdot X) \cap S = \emptyset$, a contradiction. Therefore, $S \subseteq X \setminus \{0\}$ is an $M$-m-system set.

$(\Leftarrow)$. Let $S = X \setminus P$ be an $M$-m-system in $X$. Suppose $N \cdot Z \subseteq P$, where $N$ is a submodule of $M$ and $Z$ is a submodule $X$. If $Z \not\subseteq P$ and $N \cdot X \not\subseteq P$, then $Z \cap S \neq \emptyset$ and $(N \cdot X) \cap S \neq \emptyset$. Thus $(N \cdot Z) \cap S \neq \emptyset$, a contradiction. Therefore, $P$ is an $M$-prime submodule of $X$. $\square$
Proposition 2.14. Let $X$ be an $R$-module, $P$ be a proper submodule of $X$ and $S := X \setminus P$. Then the following statements are equivalent.

(1) $P$ is an $M$-prime submodule.
(2) $S$ is an $M$-$m$-system.
(3) For every submodule $N \leq M$ and for every submodule $Z \leq X$, if $Z \cap S \neq \emptyset$ and $(N \cdot X) \cap S \neq \emptyset$, then $(N \cdot Z) \cap S \neq \emptyset$.

Proof. (1) $\Leftrightarrow$ (2) is by Corollary 2.13.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1). Suppose that $N \leq M$ and $Z \leq X$ such that $N \cdot Z \subseteq P$. If $N \cdot X \nsubseteq P$ and $Z \nsubseteq P$, then $(N \cdot X) \cap S \neq \emptyset$ and $Z \cap S \neq \emptyset$. It follows that $(N \cdot Z) \cap S \neq \emptyset$ by (3), i.e., $N \cdot Z \nsubseteq P$, a contradiction. $\square$

Proposition 2.15. Let $X$ be an $R$-module, $S \subseteq X$ be an $M$-$m$-system and $P$ be a submodule of $X$ maximal with respect to the property that $P$ is disjoint from $S$. Then $P$ is an $M$-prime submodule of $X$.

Proof. Suppose $N \cdot Z \subseteq P$, where $N \leq M$ and $Z \leq X$. If $Z \nsubseteq P$ and $N \cdot X \nsubseteq P$, then by the maximal property of $P$, we have, $(P + Z) \cap S \neq \emptyset$ and $(P + N \cdot X) \cap S \neq \emptyset$. Thus $(P + N \cdot Z) \cap S \neq \emptyset$ and it follows that $P \cap S \neq \emptyset$, a contradiction. Thus $P$ must be an $M$-prime submodule. $\square$

Next we need a generalization of the notion of $\sqrt{Y}$ for any submodule $Y$ of $X$. We adopt the following:

Definition 2.16. Let $X$ be an $R$-module. For a submodule $Y$ of $X$, if there is an $M$-prime submodule containing $Y$, then we define $$\sqrt[\downarrow]{Y} = \{x \in X : \text{every } M$-$m$-system containing } x \text{ meets } Y\}.$$ If there is no $M$-prime submodule containing $Y$, then we put $\sqrt[\downarrow]{Y} = X$.

Theorem 2.17. Let $X$ be an $R$-module and $Y \leq X$. Then either $\sqrt[\downarrow]{Y} = X$ or $\sqrt[\downarrow]{Y}$ equals the intersection of all $M$-prime submodules of $X$ containing $Y$.

Proof. Suppose that $\sqrt[\downarrow]{Y} \neq X$. This means that $$\{P : P \text{ is an } M$-$prime submodule of } X \text{ and } Y \subseteq P\} \neq \emptyset.$$ We first prove that $$\sqrt[\downarrow]{Y} \subseteq \bigcap\{P : P \text{ is an } M$-$prime submodule of } X \text{ and } Y \subseteq P\}.$$ Let $x \in \sqrt[\downarrow]{Y}$ and $P$ be any $M$-prime submodule of $X$ containing $Y$. Consider the $M$-$m$-system $X \setminus P$. This $M$-$m$-system cannot contain $x$, for otherwise it meets $Y$ and hence also $P$. Therefore, we have $x \in P$. Conversely, assume $x \notin \sqrt[\downarrow]{Y}$. Then, by Definition 2.16, there exists an $M$-$m$-system $S$ containing $x$ which is disjoint from $Y$. By Zorn’s Lemma, there exists a submodule $P \supseteq Y$ which is maximal with respect to being disjoint from $S$. By Proposition 2.15, $P$ is an $M$-prime submodule of $X$, and we have $x \notin P$, as desired. $\square$
Also, the following evident proposition offers several characterizations of $M$-semiprime modules.

**Proposition 2.18.** Let $X$ be an $R$-module. Then the following statements are equivalent.

1. $X$ is an $M$-semiprime module.
2. For every submodule $N \subseteq M$ and every submodule $Y \subseteq X$, if $N^2 \cdot Y = (0)$, then $N \cdot Y = (0)$.
3. Every nonzero submodule $Y \subseteq X$ is an $M$-semiprime module.
4. For every nonzero submodule $Y \subseteq X$, $P = \text{Ann}_M(Y)$ is a semiprime $M$-ideal.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). Suppose $0 \neq Y \subseteq X$ and $N \subseteq M$ such that $N^2 \cdot Y = (0)$. It follows that $N^2 \subseteq \text{Ann}_M(Y)$ and since $P = \text{Ann}_M(Y)$ is a semiprime $M$-ideal, there exists an $M$-semiprime module $Z$ such that $\text{Ann}_M(Y) = \text{Ann}_M(Z)$. Thus $N^2 \cdot Z = (0)$ and so $N \cdot Z = (0)$, i.e., $N \subseteq \text{Ann}_M(Z) = \text{Ann}_M(Y)$. Thus $N \cdot Y = (0)$. Therefore $X$ is an $M$-semiprime module.

**Proposition 2.19.** Let $X$ be an $R$-module. Then any intersection of $M$-semiprime submodules of $X$ is an $M$-semiprime submodule.

**Proof.** Suppose $Z_i \subseteq X$ ($i \in I$) be $M$-semiprime submodules of $X$ and put $Z = \bigcap_{i \in I} Z_i$. Suppose $Y \subseteq X$ and $N \subseteq M$ such that $N^2 \cdot Y \subseteq Z$. It follows that $N^2 \cdot Y \subseteq Z_i$ for each $i$. Since each $Z_i$ is an $M$-semiprime submodule, $N \cdot Y \subseteq Z_i$ for each $i$. Thus $N \cdot Y \subseteq Z$ and so $Z$ is an $M$-semiprime submodule.

We recall the definition of the notion of $n$-system in a ring $R$. A nonempty set $T \subseteq R$ is said to be an $n$-system set if for each $a$ in $T$, there exists $r \in R$ such that $ara \in T$ (see for example [14, Chapter 4], for more details). The complement of a semiprime ideal is an $n$-system set, and if $T$ is an $n$-system in a ring $R$ such that $a \in T$, then there exists an $n$-system $S \subseteq T$ such that $a \in S$ (see [14, Lemma 10.10]). This notion of $n$-system of rings has also generalized by Behboodi in [2] for modules. Now we have to adapt the notion of an $M$-$n$-system set to modules $R_X$.

**Definition 2.20.** Let $X$ be an $R$-module. A nonempty set $T \subseteq X \setminus \{0\}$ is called an $M$-$n$-system if, for every submodule $N \subseteq M$, and for all submodules $Y, Z \subseteq X$, if $(Y + N \cdot Z) \cap T \neq \emptyset$, then $(Y + N^2 \cdot Z) \cap T \neq \emptyset$.

**Proposition 2.21.** Let $X$ be an $R$-module. Then a submodule $P \subseteq X$ is an $M$-semiprime submodule if and only if $X \setminus P$ is an $M$-$n$-system.

**Proof.** ($\Rightarrow$). Let $T = X \setminus P$. Suppose $N$ is a submodule of $M$ and $Y, Z$ are submodules of $X$ such that $(Y + N \cdot Z) \cap T \neq \emptyset$. If $(Y + N^2 \cdot Z) \cap T = \emptyset$, then $(Y + N^2 \cdot Z) \subseteq P$. Since $P$ is $M$-semiprime submodule, $(Y + N \cdot Z) \subseteq P$. Thus $(Y + N \cdot Z) \cap T = \emptyset$, a contradiction. Therefore, $T$ is an $M$-$n$-system set in $X$. 
Then for each submodule \( Y \) any intersection of Corollary 2.25. Assume that it follows by Proposition 2.19 and Proposition 2.24. Proof. Corollary 2.26. N.

Let \( M \) be a proper submodule of \( X \) and \( T := X \setminus P \). Then the following statements are equivalent.

1. \( P \) is an \( M \)-semiprime submodule.
2. \( T \) is an \( M \)-n-system set.
3. For every submodule \( N \leq M \) and for every submodule \( Z \leq X \), if \( (N \cdot Z) \cap T \neq \emptyset \), then \( (N^2 \cdot Z) \cap T \neq \emptyset \).

Lemma 2.23 ([1, Proposition 5.6]). Assume that \( M \) is projective in \( \sigma[M] \), and let \( K, N \) be submodules of \( M \). Then \( K \cdot N \cdot X = K \cdot (N \cdot X) \) for any module \( RX \) in \( \sigma[M] \).

Proposition 2.24. Assume that \( M \) is projective in \( \sigma[M] \), and let \( X \in \sigma[M] \). Then any \( M \)-prime submodule of \( X \) is an \( M \)-semiprime submodule.

Proof. Let \( P \subseteq X \) be an \( M \)-prime submodule of \( X \) and \( N \leq M \), \( Y \subseteq X \) such that \( N^2 \cdot Y \subseteq P \). Since \( M \) is projective in \( \sigma[M] \), so \( N^2 \cdot Y = (N \cdot N) \cdot Y = N \cdot (N \cdot Y) \) by Lemma 2.23. Hence \( N \cdot (N \cdot Y) \subseteq P \). Now by assumption, \( N \cdot Y \subseteq P \) or \( N \cdot Y \subseteq P \). If \( N \cdot Y \subseteq P \), then \( P \) is an \( M \)-semiprime submodule. If \( N \cdot Y \subseteq P \), then \( N \cdot Y \subseteq N \cdot X \subseteq P \). Thus \( P \) is an \( M \)-semiprime submodule. □

Corollary 2.25. Assume that \( M \) is projective in \( \sigma[M] \) and \( X \in \sigma[M] \). Then any intersection of \( M \)-prime submodules of \( X \) is an \( M \)-semiprime submodule.

Proof. It follows by Proposition 2.19 and Proposition 2.24. □

Corollary 2.26. Assume that \( M \) is projective in \( \sigma[M] \), and let \( X \in \sigma[M] \). Then for each submodule \( Y \) of \( X \), either \( \sqrt{Y} = X \) or \( \sqrt{Y} \) is an \( M \)-semiprime submodule of \( X \).

Proof. By Theorem 2.17 and Corollary 2.25, it is clear. □

Definition 2.27. Let \( M \) be an \( R \)-module. For any module \( X \), we define \( \text{rad}_M(X) = \sqrt[\mathbb{Q}(0)]{0} \). This is called \( M \)-Baer-McCoy radical or \( M \)-prime radical of \( X \). Thus if \( X \) has an \( M \)-prime submodule, then \( \text{rad}_M(X) \) is equal to the intersection of all the \( M \)-prime submodules in \( X \) but, if \( X \) has no \( M \)-prime submodule, then \( \text{rad}_M(X) = X \).

The following two propositions have been established in [2] for prime radical of modules. Now by the same method as [2], we extend these facts to \( M \)-prime radical of modules.

Proposition 2.28. Let \( X \) be an \( R \)-module and \( Y \subseteq X \). Then \( \text{rad}_M(Y) \subseteq \text{rad}_M(X) \).
Lemma 2.29. Assume that $\sigma$ in follows that rad $Y$ $\subseteq Y \cap P \subseteq P$. Thus in any case, rad$_M(Y)$ $\subseteq P$. It follows that rad$_M(Y)$ $\subseteq$ rad$_M(X)$.

Proof. Let $P$ be any M-prime submodule of $X$. If $Y \subseteq P$, then rad$_M(Y)$ $\subseteq P$. If $Y \not\subseteq P$, then it is easy to check that $Y \cap P$ is an M-prime submodule of $Y$, and hence rad$_M(Y)$ $\subseteq (Y \cap P)$ $\subseteq P$. Thus in any case, rad$_M(Y)$ $\subseteq P$. It follows that rad$_M(Y)$ $\subseteq$ rad$_M(X)$.

Lemma 2.29. Assume that $M$ is projective in $\sigma[M]$, and let $X$ be an R-module in $\sigma[M]$ such that $X = \bigoplus_{\lambda \in \Lambda} X_\lambda$ is a direct sum of submodules $X_\lambda$ ($\lambda \in \Lambda$). Then for every submodule $N \subseteq M$, we have

$$N \cdot X = \bigoplus_{\lambda \in \Lambda} N \cdot X_\lambda.$$  

Proof. Since for every $\lambda \in \Lambda$, $X_\lambda \subseteq X$, $N \cdot X_\lambda \subseteq N \cdot X$ for every $\lambda \in \Lambda$. It follows that $\bigoplus_{\lambda \in \Lambda} N \cdot X_\lambda \subseteq N \cdot X$. On the other hand, since $M$ is projective in $\sigma[M]$, so $N \cdot X = \sum_{f \in \text{Hom}_R(M,X)} f(N)$ and for every $\lambda \in \Lambda$, $N \cdot X_\lambda = \sum_{f \in \text{Hom}_R(M,X_\lambda)} f(N)$ by Lemma 2.9 (a). Now let $x \in N \cdot X$. Thus $x = \sum_{t=1}^t f_t(n_t)$ where $t \in \mathbb{N}$, $n_t \in N$ and $f_t \in \text{Hom}_R(M,X)$. Since $f_t(n_t) \in X$, so for every $1 \leq t \leq t$, $f_t(n_t) = \{x_t^{(i)}\}$, where $x_t^{(i)} \in X_\lambda$. Thus $x = \{x_\lambda^{(1)} + \cdots + x_\lambda^{(t)}\} = \{\pi_\lambda f_1(n_1) + \cdots + \pi_\lambda f_t(n_t)\}$, where $\pi_\lambda : X \rightarrow X_\lambda$ is the canonical projection for every $\lambda \in \Lambda$. It is clear that by Lemma 2.9, $\sum_{i=1}^t \pi_\lambda f_i(n_i) \in N \cdot X_\lambda$ for every $\lambda \in \Lambda$. Thus $x \in \bigoplus_{\lambda \in \Lambda} N \cdot X_\lambda$.

We note that, since in Lemma 2.29, we assume that $M$ is projective in $\sigma[M]$, so our product coincides with the product defined in [6, Definition 1.1]. Thus Lemma 2.29 is also proved in [6, Proposition 1.3 (8)].

Proposition 2.30. Assume that $M$ is projective in $\sigma[M]$, and let $X$ be an R-module in $\sigma[M]$ such that $X = \bigoplus_{\lambda \in \Lambda} X_\lambda$ is a direct sum of submodules $X_\lambda$ ($\lambda \in \Lambda$). Then

$$\text{rad}_M(X) = \bigoplus_{\lambda \in \Lambda} \text{rad}_M(X_\lambda).$$

Proof. By Proposition 2.28, $\text{rad}_M(X_\lambda) \subseteq \text{rad}_M(X)$ for all $\lambda \in \Lambda$. Thus $\bigoplus_{\lambda \in \Lambda} \text{rad}_M(X_\lambda) \subseteq \text{rad}_M(X)$. Now let $x \notin \bigoplus_{\lambda \in \Lambda} \text{rad}_M(X_\lambda)$, for some $x \in X$. Then there exists $\mu \in \Lambda$ such that $\pi_\mu(x) \notin \text{rad}_M(X_\mu)$, where $\pi_\mu : X \rightarrow X_\mu$ denotes the canonical projection. Thus there exists an M-prime submodule $Y_\mu$ of $X_\mu$ such that $\pi_\mu(x) \notin Y_\mu$. Let $Z = Y_\mu \bigoplus_{\lambda \neq \mu} X_\lambda$. It is easy to check by Lemma 2.29 that $Z$ is an M-prime submodule of $X$ and $x \notin Z$. Thus $x \notin \text{rad}_M(X)$. It follows that $\text{rad}_M(X) \subseteq \bigoplus_{\lambda \in \Lambda} \text{rad}_M(X_\lambda)$.

3. M-Baer’s lower nilradical of modules

We recall the definition of the nilpotent element in a module. An element $x$ of an R-module $X$ is called nilpotent if $x = \sum_{i=1}^r a_i x_i$ for some $a_i \in R$, $x_i \in X$ and $r \in \mathbb{N}$, such that $a_i^k x_i = 0$ ($1 \leq i \leq r$) for some $k \in \mathbb{N}$ and $x$ is called strongly nilpotent if $x = \sum_{i=1}^r a_i x_i$, for some $a_i \in R$, $x_i \in X$ and $r \in \mathbb{N}$, such that for every $1 \leq i \leq r$ and every sequence $a_{i1}, a_{i2}, a_{i3}, \ldots$, where $a_{i1} = a_i$,
and \( a_{m+1} \in a_m R a_m (\forall n) \), we have \( a_{k} R x_i = 0 \) for some \( k \in \mathbb{N} \) (see [4]). It is clear that every strongly nilpotent element of a module \( X \) is a nilpotent element but the converse is not true (see the example 2.3 [4]). In case that \( R \) is commutative ring, nilpotent and strongly nilpotent are equal.

This notion has been generalized to modules over a projective module \( M \) in \( \sigma[M] \).

**Definition 3.1.** Assume that \( M \) is projective in \( \sigma[M] \), and let \( X \) be an \( R \)-module in \( \sigma[M] \). Then an element \( x \in X \) is called \( M \)-nilpotent if \( x = \sum_{i=1}^{n} r_i f_i(m_i) \) for some \( r_i \in R, m_i \in M, n \in \mathbb{N} \) and \( f_i \in \text{Hom}_R(M, R x_i) \), where \( x_i \in X \) such that \( r_i^k f_i(m_i) = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \). Also, an element \( x \in X \) is called **strongly \( M \)-nilpotent** if \( x = \sum_{i=1}^{n} r_i f_i(m_i) \) for some \( r_i \in R, m_i \in M, n \in \mathbb{N} \) and \( f_i \in \text{Hom}_R(M, R x_i) \), where \( x_i \in X \) such that for every \( i (1 \leq i \leq n) \) and every sequence \( r_{i1}, r_{i2}, r_{i3}, \ldots \), where \( r_{i1} = r_i \) and \( r_{it+1} \in r_i R r_{it} (\forall t) \), we have \( r_{ik} R f_i(m_i) = 0 \) for some \( k \in \mathbb{N} \).

**Proposition 3.2.** Let \( X \) be an \( R \)-module. Then an element \( x \in X \) is strongly \( R \)-nilpotent if and only if \( x \) is strongly \( R \)-nilpotent.

**Proof.** \((\Rightarrow)\). Suppose that \( x \in X \) is strongly nilpotent. Then \( x = \sum_{i=1}^{n} r_{i} x_i \) for some \( r_{i} \in R, x_{i} \in X, n \in \mathbb{N} \) such that for every \( i (1 \leq i \leq n) \) and for every sequence \( r_{i1}, r_{i2}, r_{i3}, \ldots \), where \( r_{i1} = r_i \) and \( r_{it+1} \in r_i R r_{it} (\forall t) \), we have \( r_{ik} R x_i = 0 \) for some \( k \in \mathbb{N} \). Now consider \( f_i : R \to R x_i \) such that \( f_i(r) = r x_i \). Then \( f_i(1) = x_i \) and it follows that \( x = \sum_{i=1}^{n} r_{i} x_i = \sum_{i=1}^{n} r_{i} f_i(1) \). Since \( r_{ik} R x_i = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \), we conclude that \( r_{ik} R f_i(1) = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \), i.e., \( x \) is a strongly \( R \)-nilpotent element of \( X \).

\((\Leftarrow)\). Assume that \( x \in X \) is strongly \( R \)-nilpotent. Thus \( x = \sum_{i=1}^{n} r_{i} f_i(a_i) \) for some \( r_{i}, a_i \in R, n \in \mathbb{N} \) and \( f_i \in \text{Hom}_R(R, R x_i) \), where \( x_i \in X \) such that for every \( i (1 \leq i \leq n) \) and for every sequence \( r_{i1}, r_{i2}, r_{i3}, \ldots \), where \( r_{i1} = r_i \) and \( r_{it+1} \in r_i R r_{it} (\forall t) \), we have \( r_{ik} R f_i(a_i) = 0 \) for some \( k \in \mathbb{N} \). Since \( f_i(a_i) \in R x_i \subseteq X \), we conclude that \( x \) is a strongly nilpotent element of \( X \). \(\square\)

**Proposition 3.3.** Let \( X \) be an \( R \)-module. Then an element \( x \in X \) is nilpotent if and only if \( x \) is \( R \)-nilpotent.

**Proof.** \((\Rightarrow)\). Assume that \( x \in X \) is nilpotent. Thus \( x = \sum_{i=1}^{n} r_{i} x_i \) for some \( r_{i} \in R, x_{i} \in X, n \in \mathbb{N} \) such that \( r_{i}^k x_i = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \). Now consider \( f_i : R \to R x_i \) such that \( f_i(r) = r x_i \), so \( f_i(1) = x_i \). It follows that \( x = \sum_{i=1}^{n} r_{i} x_i = \sum_{i=1}^{n} r_{i} f_i(1) \). Since \( r_{i}^k x_i = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \), so \( r_{i}^k f_i(1) = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \), i.e., \( x \) is an \( R \)-nilpotent element of \( X \).

\((\Leftarrow)\). Assume that \( x \in X \) is an \( R \)-nilpotent element. Thus \( x = \sum_{i=1}^{n} r_{i} f_i(a_i) \) for some \( r_{i}, a_i \in R, n \in \mathbb{N} \) and \( f_i \in \text{Hom}_R(R, R x_i) \), where \( x_i \in X \) such that \( r_{i}^k f_i(a_i) = 0 (1 \leq i \leq n) \) for some \( k \in \mathbb{N} \). Since \( f_i(a_i) \in R x_i \subseteq X \), we conclude that \( x \) is a nilpotent element of \( X \). \(\square\)
Proposition 3.4. Assume that $R$ is a commutative ring, $M$ is projective in $\sigma[M]$ and $X \in \sigma[M]$. Then an element $x \in X$ is $M$-nilpotent if and only if $x$ is strongly $M$-nilpotent.

Proof. ($\Rightarrow$). Assume that $x \in X$ is $M$-nilpotent. Thus $x = \sum_{i=1}^{n} r_{i}f_{i}(m_{i})$ for some $r_{i} \in R$, $m_{i} \in M$, $n \in \mathbb{N}$ and $f_{i} \in \text{Hom}_{R}(M,Rx_{i})$, where $x_{i} \in X$ such that $r_{i}^{k}f_{i}(m_{i}) = 0$ (for some $1 \leq i \leq n$) for some $k \in \mathbb{N}$. Consider sequence $r_{i1},r_{i2},r_{i3},\ldots$, where $r_{i1} = r_{i}$ and $r_{it+1} = r_{it}Rx_{it}$ for every $1 \leq i \leq n$ and $(\forall t)$. Thus there exists an element $r_{ik} = r_{i1}^{k}r'$ (where $r' \in R$) such that $r_{ik}Rf_{i}(m_{i}) = r_{i1}^{k}r'Rf_{i}(m_{i}) = 0$ (since $R$ is commutative and $r_{i1}^{k}f_{i}(m_{i}) = 0$). Thus $x \in X$ is a strongly $M$-nilpotent element.

($\Leftarrow$). Suppose that $x \in X$ is a strongly $M$-nilpotent element. Thus $x = \sum_{i=1}^{n} r_{i}f_{i}(m_{i})$ for some $r_{i} \in R$, $m_{i} \in M$, $n \in \mathbb{N}$ and $f_{i} \in \text{Hom}_{R}(M,Rx_{i})$, where $x_{i} \in X$ such that for every $i (1 \leq i \leq n)$ and for every sequence $r_{i1},r_{i2},r_{i3},\ldots$, where $r_{i1} = r_{i}$ and $r_{it+1} = r_{it}Rx_{it}$ we have $r_{ik}Rf_{i}(m_{i}) = 0$ for some $k \in \mathbb{N}$. Consider sequence $r_{i1},r_{i2},r_{i3},\ldots$, where $r_{i1} = r_{i}$ and $r_{i2} = r_{i1}^{2} = r_{i1}r_{i1}$, $r_{i3} = r_{i1}^{3} = r_{i1}r_{i1}r_{i1}$, $r_{i4} = r_{i1}r_{i1}r_{i1}r_{i1}$, $r_{i5} = r_{i1}r_{i1}r_{i1}r_{i1}r_{i1}$, $\ldots$. By assumption, we have $r_{ik}Rf_{i}(m_{i}) = 0$ for some $k \in \mathbb{N}$. Since $r_{ik} = r_{i1}^{k}$ for some $k \in \mathbb{N}$, so $r_{i1}^{k}Rf_{i}(m_{i}) = r_{ik}Rf_{i}(m_{i}) = 0$. Now for $r = 1$, we have $r_{i1}^{k}1f_{i}(m_{i}) = 0$. Thus $x$ is an $M$-nilpotent element.

We recall the definition of Baer’s lower nilradical in a module. For any module $X$, $\text{Nil}_{s}(R \times X)$ is the set of all strongly nilpotent elements of $X$. In case that $R$ is a commutative ring, $\text{Nil}_{s}(R \times X)$ is the set of all nilpotent elements of $X$.

Definition 3.5. Assume that $M$ is projective in $\sigma[M]$. For any module $X$ in $\sigma[M]$, we define $M-\text{Nil}_{s}(R \times X)$ to be the set of all strongly $M$-nilpotent elements of $X$. This is called $M$-Baer’s lower nilradical of $X$.

Proposition 3.6. Assume that $M$ is projective in $\sigma[M]$. Then for any module $X$ in $\sigma[M]$ $$(\forall x \in \text{Nil}_{s}(M) \cdot X \subseteq M - \text{Nil}_{s}(R \times X) \subseteq \text{rad}_{M}(X).$$

Proof. Since $M$ is projective in $\sigma[M]$, by Lemma 2.9 (a), $$(\forall x \in \text{Nil}_{s}(M) \cdot X = \sum_{f \in \text{Hom}_{M}(M,X)} f(\text{Nil}_{s}(M)).$$

Now let $x \in \text{Nil}_{s}(M) \cdot X$. Thus $x = \sum_{s=1}^{s} f_{s}(m_{s})$ for some $m_{s} \in \text{Nil}_{s}(M)$, $s \in (N)$ and $f_{s} \in \text{Hom}_{R}(M,X)$. Since $m_{s} \in \text{Nil}_{s}(M)$, so $m_{s} = \sum_{j=1}^{t} r_{ij}n_{ij}$ for some $r_{ij} \in R$, $n_{ij} \in M$, $t \in \mathbb{N}$ such that for every $j (1 \leq j \leq t)$ and for every sequence $r_{ij1},r_{ij2},r_{ij3},\ldots$, where $r_{ij1} = r_{ij}$ and $r_{ijt+1} = r_{ij}Rx_{it}$ $(\forall t)$, we have $r_{ijk}Rn_{ij} = 0$ for some $k \in \mathbb{N}$. Thus $x = \sum_{s=1}^{s} f_{s}(\sum_{j=1}^{s} r_{ij}n_{ij}) = \sum_{s=1}^{s} \sum_{j=1}^{t} r_{ij}f_{s}(n_{ij})$. Since $r_{ijk}Rn_{ij} = 0$, we conclude that $0 = f_{s}(r_{ijk}Rn_{ij}) = r_{ijk}Rf_{s}(n_{ij})$ for some $k \in \mathbb{N}$, where $(1 \leq i \leq s)$ and $(1 \leq j \leq t)$. Thus $x \in M - \text{Nil}_{s}(R \times X)$. 


Let $x \in M\text{-}\text{Nil}_s(R X)$ and $x \notin \text{rad}_M(X) = \sqrt{\{0\}}$. So $x = \sum_{i=1}^{n} a_i f_i(m_i)$ for some $a_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \text{Hom}_R(M, R x)$ such that for every $i (1 \leq i \leq n)$ and for every sequence $a_{i1}, a_{i2}, a_{i3}, \ldots$, where $a_{i1} = a_i$ and $a_{iu+1} = a_{iu} R a_u (\forall u)$, we have $a_{ik} R f_i(m_i) = 0$ for some $k \in \mathbb{N}$. Without loss of generality, we can assume that $a_1 f_1(m_1) \notin \text{rad}_M(X)$. Thus there exists an $M$-m-system $S$ such that $a_1 f_1(m_1) \in S$ and $0 \notin S$. On the other hand $a_1 f_1(m_1) \in R a_1(R m_1) \cdot (R x_1)$. Thus $R a_1(R m_1) \cdot (R x_1) \cap S \neq \emptyset$ and hence $R a_1(R m_1) \cdot X \cap S \neq \emptyset$. Therefore, if we put $N = R a_1(R m_1)$, $Y = (0)$ and $Z = R a_1(R m_1) \cdot (R x_1)$, then $(R a_1(R m_1))^2 \cdot (R x_1) \cap S \neq \emptyset$ by Proposition 2.14. Since $M$ is projective in $\sigma[M]$, by Lemma 2.9(a) and Lemma 2.23, we conclude that

$$
(R a_1(R m_1))^2 \cdot (R x_1) = (R a_1(R m_1) \cdot R a_1(R m_1)) \cdot (R x_1) = (R a_1(R m_1)) \cdot (R a_1(R m_1) \cdot (R x_1)) = \sum_{f \in \text{Hom}_R(M, R a_1(R m_1) \cdot (R x_1))} f(R a_1(R m_1)).
$$

Assume that $s_1 = 1$, $a_{11} = a_1$ and $a_{11} f_1(t_1 a_{12} s_2 m_1) \in (R a_1(R m_1))^2 \cdot (R x_1) \cap S$, where $s_2, t_1 \in R$. Since $a_1 f_1(t_1 a_{12} s_2 m_1) = s_2 a_{11} t_1 a_{12} f_1(m_1)$ and $a_{12} = a_1 t_1 a_{11}$, so $s_2 a_{12} f_1(m_1) \in R a_{12}(R m_1) \cdot (R x_1) \cap S$. It follows that $R a_{12}(R m_1) \cdot (R x_1) \cap S \neq \emptyset$ and so

$$(R a_{12}(R m_1))^2 \cdot (R x_1) \cap S \neq \emptyset.$$ 

Thus there exists $s_3 a_{13} f_1(m_1) \in (R a_{12}(R m_1))^2 \cdot (R x_1) \cap S$, where $s_3 \in R$, and $a_{13} := a_{12} t_2 s_2 a_{12}$ for some $t_2 \in R$. We can repeat this argument to get sequences $\{s_u\}_{u \in \mathbb{N}}$ and $\{a_{1u}\}_{u \in \mathbb{N}}$ in $R$, where $a_{11} = a_1$ and $a_{1u+1} \in a_{1u} R a_u (\forall u)$, such that $s_u a_{1u} f_1(m_1) \in S$ for all $u \geq 1$. Now by our hypothesis $a_k R f_1(m_1) = 0$ for some $k \in \mathbb{N}$, and so $s_k a_{1k} f_1(m_1) = 0 \in S$, a contradiction. 

In case $M = R$, by Proposition 3.6, $\text{Nil}_s(R) \cdot X \subseteq R\text{-}\text{Nil}_s(R X) \subseteq \text{rad}_R(X)$. Since by Proposition 3.2, $R\text{-}\text{Nil}_s(R X)$ is the set of all strongly $R$-nilpotent elements of $X$, so we have $R\text{-}\text{Nil}_s(R X) = \text{Nil}_s(R X)$ (see also, [2, Lemma 3.2]).

**Corollary 3.7.** Assume that $M$ is projective in $\sigma[M]$. Then

$$\text{Nil}_s(M) = \text{Nil}_s(M) \cdot M = M - \text{Nil}_s(M).$$

**Proof.** By Proposition 3.6, $\text{Nil}_s(M) \cdot M \subseteq M\text{-}\text{Nil}_s(M)$. Also, we have $\text{Nil}_s(M) \cdot M = \sum_{f \in \text{Hom}_R(M, M)} f(\text{Nil}_s(M))$, by Lemma 2.9(a). Since $1_M \in \text{Hom}_R(M, M)$, so $\text{Nil}_s(M) \subseteq \text{Nil}_s(M) \cdot M$. On the other hand, if $x \in M\text{-}\text{Nil}_s(M)$, then $x = \sum_{i=1}^{n} r_i f_i(m_i)$ for some $r_i \in R$, $m_i \in M$, $n \in \mathbb{N}$ and $f_i \in \text{Hom}_R(M, R x_i)$, where $x_i \in M$ such that for every $i (1 \leq i \leq n)$ and for every sequence $r_{i1}, r_{i2}, r_{i3}, \ldots$, where $r_{i1} = r_i$ and $r_{i+1} \in r_i R r_i (\forall i)$, we have $r_{ik} R f_i(m_i) = 0$ for some $k \in \mathbb{N}$. Since $f_i(m_i) \in R x_i \subseteq R x$, it follows that $x$ is a strongly nilpotent element of $M$. So $x \in \text{Nil}_s(M)$. It follows that $M\text{-}\text{Nil}_s(M) \subseteq \text{Nil}_s(M)$.
and \( \text{Nil}_a(M) \subseteq \text{Nil}_a(M) \cdot M \subseteq M \)-\( \text{Nil}_a(M) \subseteq \text{Nil}_a(M) \). Thus \( \text{Nil}_a(M) = \text{Nil}_a(M) \cdot M = M \)-\( \text{Nil}_a(M) \).

\( \square \)

**Corollary 3.8.** Assume that \( M \) is projective in \( \sigma[M] \). Then \( \text{rad}_R(M) \subseteq \text{rad}_M(M) \).

**Proof.** By Proposition 3.6, we have \( M \)-\( \text{Nil}_a(M) \subseteq \text{rad}_M(M) \). On the other hand \( \text{Nil}_a(M) = M \)-\( \text{Nil}_a(M) \) by Corollary 3.7. Thus \( \text{Nil}_a(M) \subseteq \text{rad}_M(M) \).

Since \( M \) is projective in \( \sigma[M] \), \( \text{rad}_R(M) = \text{Nil}_a(M) \) by [2, Theorem 3.8]. Thus \( \text{rad}_R(M) = \text{Nil}_a(M) \subseteq \text{rad}_M(M) \). \( \square \)

**Proposition 3.9.** Assume that \( M \) is projective in \( \sigma[M] \). If \( X \in \sigma[M] \) such that \( \text{rad}_M(X) = M \)-\( \text{Nil}_a(X) \), then \( \text{rad}_M(Y) = M \)-\( \text{Nil}_a(Y) \) for any direct summand \( Y \) of \( X \).

**Proof.** Suppose that \( X = Y \oplus Z \), where \( Z, Y \) are submodules of \( X \). By Proposition 3.6, \( M \)-\( \text{Nil}_a(Y) \subseteq \text{rad}_M(Y) \). Let \( x \in \text{rad}_M(Y) \). By Proposition 2.28, \( x \in \text{rad}_M(X) \). By hypothesis \( x \in M \)-\( \text{Nil}_a(X) \). Thus \( x = \sum_{i=1}^{n} r_i f_i(m_i) \) for some \( r_i \in R, m_i \in M, n \in \mathbb{N} \) and \( f_i \in \text{Hom}_R(M, Rx_i) \), where \( x_i \in X \) such that for every \( i (1 \leq i \leq n) \) and for every sequence \( r_1, r_2, r_3, \ldots \), where \( r_1 = r_i \) and \( r_{i+1} \in r_i R x_i \) \((\forall i)\), we have \( r_i R f_i(m_i) = 0 \) for some \( k \in \mathbb{N} \).

Since \( x_i \in X \), there exist elements \( y_i \in Y, z_i \in Z \) such that \( x_i = y_i + z_i \) for each \( i (1 \leq i \leq n) \). On the other hand, \( f_i(m_i) \in Rx_i \) for each \( i \), and hence \( f_i(m_i) = a_i(y_i + z_i) \) for some \( a_i \in R (1 \leq i \leq n) \). It is clear that \( x = r_1 a_1 y_1 + r_2 a_2 y_2 + \cdots + r_n a_n y_n \), and \( r_{ik} R a_i y_i = 0 \) for some \( k \in \mathbb{N} (1 \leq i \leq n) \). Now for each \( i (1 \leq i \leq n) \), we consider \( g_i : M \xrightarrow{\pi_i} Rx_i \subseteq X \xrightarrow{\pi_i} Ry_i \subseteq Y \), where \( \pi_i \) is the natural projection map such that \( g_i(m_i) = \pi_i f_i(m_i) = \pi_i(a_i(y_i + z_i)) = a_i y_i \). Thus \( x = r_1 a_1 y_1 + r_2 a_2 y_2 + \cdots + r_n a_n y_n = \sum_{i=1}^{n} r_i g_i(m_i) \), where \( g_i \in \text{Hom}_R(M, Ry_i) \) and \( r_{ik} R a_i y_i = r_{ik} R g_i(m_i) = 0 \). It follows that \( x \in M \)-\( \text{Nil}_a(Y) \). Thus \( \text{rad}_M(Y) = M \)-\( \text{Nil}_a(Y) \). \( \square \)

4. **M-injective modules and prime M-ideals**

The module \( R X \) is said to be \( M \)-generated if there exists an \( R \)-epimorphism from a direct sum of copies of \( M \) onto \( X \). Equivalently, for each nonzero \( R \)-homomorphism \( f : X \rightarrow Y \) there exists an \( R \)-homomorphism \( g : M \rightarrow X \) with \( fg \neq 0 \). The trace of \( M \) in \( X \) is defined to be

\[
\text{tr}^M(X) = \sum_{f \in \text{Hom}_R(M, X)} f(M)
\]

and thus \( X \) is \( M \)-generated if and only if \( \text{tr}^M(X) = X \).

We recall the definition of prime \( M \)-ideal. The proper \( M \)-ideal \( P \) is said to be a prime \( M \)-ideal if there exists an \( M \)-prime module \( R X \) such that \( P = \text{Ann}_M(X) \).
Proposition 4.1. Let $M$ an $R$-module with $\text{Hom}_R(M,X) \neq 0$ for every $X \in \sigma[M]$ and $P$ be a proper $M$-ideal. Then $P$ is a prime $M$-ideal if and only if $P$ is a Beachy-prime $M$-ideal.

Proof. Assume that $P$ is a prime $M$-ideal. Thus there exists $M$-prime module $X$ such that $P = \text{Ann}_M(X)$. Since $P \neq M$, $\text{Hom}_R(M,X) \neq 0$. Thus by Proposition 2.7, $X$ is a Beachy-M-prime module. Thus $P$ is a Beachy-prime $M$-ideal.

Conversely, let $P$ be a Beachy-prime $M$-ideal. Thus there exists a Beachy-M-prime module $X$ in $\sigma[M]$ such that $P = \text{Ann}_M(X)$. Since $\text{Hom}_R(M,X) \neq 0$, so $X \neq (0)$. Now assume that $Y$ is a nonzero submodule of $X$. So $Y \in \sigma[M]$ and $\text{Hom}_R(M,Y) \neq 0$ by assumption. Therefore, $\text{Ann}_M(X) = \text{Ann}_M(Y)$ by the definition of Beachy-M-prime module. Thus by Proposition 2.4, $X$ is an $M$-prime module and hence $P$ is a prime $M$-ideal. □

The module $rX$ in $\sigma[M]$ is said to be \textit{finitely $M$-generated} if there exists an epimorphism $f : M^n \to X$, for some positive integer $n$. It is said to be \textit{finitely $M$-annihilated} if there exists a monomorphism $g : M/\text{Ann}_M(X) \to X^m$, for some positive integer $m$. Also, the module $rM$ is said to satisfy \textit{condition H} if every finitely $M$-generated module is finitely $M$-annihilated. Note that if $M = R$ and $R$ is a fully bounded Noetherian ring, then $M$ satisfies condition $H$. The same is true if $M$ is an Artinian module, since then $M/K$ has the finite intersection property.

In [1, Theorem 6.7], it is shown that if $M$ is a Noetherian module such that $M$ satisfies condition $H$ and $\text{Hom}_R(M,X) \neq 0$ for all modules $X$ in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable $M$-injective modules in $\sigma[M]$ and Beachy-prime $M$-ideals.

Next, in the main result of this section, we show this fact is also true for a Noetherian module with condition $H$ and the assumption $\text{Hom}_R(M,X) \neq 0$ for all modules $X$ in $\sigma[M]$ via prime $M$-ideals.

Corollary 4.2. Let $M$ be a Noetherian $R$-module. If $M$ satisfies condition $H$ and $\text{Hom}_R(M,X) \neq 0$ for all modules $X$ in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable $M$-injective modules in $\sigma[M]$ and prime $M$-ideals.

Proof. By [1, Theorem 6.7] and Proposition 4.1, it is clear. □

5. Prime $M$-ideals and $M$-prime radical of Artinian modules

Let $M$ be an $R$-module. Recall that a proper submodule $P$ of $M$ is \textit{virtually maximal} if the factor module $M/P$ is a homogeneous semisimple $R$-module, i.e., $M/P$ is a direct sum of isomorphic simple modules. Clearly, every virtually maximal submodule of $M$ is prime. Also, every maximal submodule of $M$ is virtually maximal and for $M = R$ and $R$ commutative, this is equivalent to the notion of maximal ideal in $R$. 

We recall that $\text{Soc}(M)$ is sum of all minimal submodules of $M$. If $M$ has no minimal submodule, then $\text{Soc}(M) = (0)$.

**Proposition 5.1.** Let $M$ be an Artinian $R$-module. If $M$ is an $M$-prime module, then $M$ is a homogeneous semisimple module.

*Proof.* Since $M$ is an Artinian $R$-module, $\text{Soc}(M) \neq (0)$. Hence there exist simple submodule $Rm$ of $M$ where $0 \neq m \in M$. Since $M$ is an $M$-prime module, $\text{Ann}_M(Rm) = \text{Ann}_M(M) = (0)$ by Proposition 2.4. Thus $(0) = \text{Ann}_M(Rm) = \bigcap_{f \in \text{Hom}_R(M,Rm)} \ker(f)$. Since $Rm \cong M/\ker(f)$ for every $f \in \text{Hom}_R(M,Rm)$, $(0)$ is an intersection of maximal submodules and since $M$ is Artinian, $(0)$ must be a finite intersection of maximal submodules. It follows that $M$ is isomorphic to a finite direct sum of copies of $Rm$. Thus $M$ is a homogeneous semisimple module. □

An $M$-ideal $P$ is said to be a **primitive $M$-ideal** if $P = \text{Ann}_M(S)$ for a simple $R$-module $S$ (see [1, Definition 3.5]).

**Proposition 5.2.** Let $P$ be a proper $M$-ideal. If $P$ is a primitive $M$-ideal, then $P$ is a prime $M$-ideal.

*Proof.* If $P$ is a primitive $M$-ideal, then $P = \text{Ann}_M(S)$ for a simple $R$-module $S$. Since $S$ has no any proper submodule, $S$ is an $M$-prime module by Proposition 2.4. Thus $P$ is a prime $M$-ideal. □

**Proposition 5.3.** Let $M$ be an $M$-prime module with $\text{Soc}(M) \neq (0)$. Then $(0)$ is a primitive $M$-ideal.

*Proof.* Since $\text{Soc}(M) \neq (0)$, there exists a simple submodule $Rm$ of $M$ where $0 \neq m \in M$. Since $M$ is an $M$-prime module, so $\text{Ann}_M(Rm) = \text{Ann}_M(M) = (0)$. Therefore, $(0)$ is a primitive $M$-ideal. □

**Proposition 5.4.** Assume that $M$ is projective in $\sigma[M]$. If $M$ is an Artinian $R$-module, then every prime $M$-ideal of $M$ is virtually maximal.

*Proof.* Suppose that $P \preceq M$ is a prime $M$-ideal. Since $M$ is projective in $\sigma[M]$, $M/P$ is an $M$-prime module by Proposition 2.10. Since $M/P$ is also an Artinian module, $\text{Soc}(M/P) \neq (0)$ and hence there exists a simple submodule $Rm$ of $M/P$ where $0 \neq \bar{m} \in M/P$. Since $M/P$ is an $M$-prime module, $\text{Ann}_M(Rm) = \text{Ann}_M(M/P) = P$. On the other hand, $P = \text{Ann}_M(Rm) = \bigcap_{f \in \text{Hom}_R(M,Rm)} \ker(f)$. Since $Rm \cong M/\ker(f)$ for every $f \in \text{Hom}_R(M,Rm)$, $P$ must be an intersection of maximal submodules. Since $M/P$ is Artinian, $P$ must be a finite intersection of maximal submodules, and so $M/P$ is isomorphic to a finite direct sum of copies of $Rm$. Thus $M/P$ is a homogeneous semisimple module, i.e., $P$ is a virtually maximal submodule of $M$. □

**Definition 5.5.** The **prime radical** of the module $M$, denoted by $P(M)$, is defined to be the intersection of all prime $M$-ideals.
We note that each prime $M$-ideal is the annihilator of an $M$-prime module in $M$. It follows that $P(M) = \text{rad}_{C}(M)$, where $C$ is the class of all $M$-prime left $R$-modules. If $RX$ is any module with a submodule $Y$ such that $X/Y$ is an $M$-prime module, then $\text{rad}_{C}(X) \subseteq Y$. In this case it follows from [1, Lemma 1.8] that $P(M) \cdot X \subseteq Y$.

**Theorem 5.6.** Assume that $M$ is projective in $\sigma[M]$. If $M$ is an Artinian $R$-module, then every prime $M$-ideal of $M$ is virtually maximal and $M/P(M)$ is a Noetherian $R$-module.

**Proof.** If $M$ does not contain any prime $M$-ideal, then $P(M) = M$. Suppose that $M$ contains a prime $M$-ideal. By Proposition 5.4, every prime $M$-ideal of $M$ is virtually maximal. Let $N$ be minimal in the collection $S$ of $M$-ideals of $M$ which are finite intersections of primes. If $P$ is any prime $M$-ideal of $M$, then $P \cap N \in S$ and $P \cap N \subseteq N$. Thus $N = P \cap N \subseteq P$ by minimality of $N$ in $S$. It follows that $N = P(M)$. On the other hand, for each prime $M$-ideal, the factor module $M/P$ is a homogeneous semisimple module with DCC. So $M/P$ is Noetherian. Thus $M/P$ is Noetherian for every prime $M$-ideal $P$ of $M$. Since $P(M)$ is a finite intersection of prime $M$-ideals, $M/P(M)$ is also a Noetherian $R$-module. □

The following theorem is a generalization of [2, Theorem 2.11].

**Theorem 5.7.** Assume that $M$ is projective in $\sigma[M]$. If $M$ be an Artinian $R$-module, then $P(M) = M$ or there exist primitive $M$-ideals $P_1, \ldots, P_n$ of $M$ such that $P(M) = \bigcap_{i=1}^{n} P_i$.

**Proof.** Let $P$ be a prime $M$-ideal of $M$. Since $M$ is projective in $\sigma[M]$, so $M/P$ is an $M$-prime module by Proposition 2.10 (ii). Since $M/P$ is an Artinian $R$-module, $\text{Soc}(M/P) \neq (0)$. Thus there exists a simple submodule $R\bar{m}$ of $M/P$ where $0 \neq \bar{m} \in M/P$. Since $M/P$ is an $M$-prime module, $\text{Ann}_M(R\bar{m}) = \text{Ann}_M(M/P)$. On the other hand, $\text{Ann}_M(M/P) = P$, since $P$ is an $M$-ideal. Thus $P$ is a primitive $M$-ideal. Since $P$ is arbitrary prime $M$-ideal, so every prime $M$-ideal of $M$ is primitive $M$-ideal. On the other hand by Proposition 5.2, we have that every primitive $M$-ideals is prime $M$-ideal. Thus $P(M)$ is the intersection all of primitive $M$-ideal of $M$. Now let $N$ be minimal in the collection $S$ of $M$-ideals of $M$ which are finite intersections of primes. If $Q$ is any prime $M$-ideal of $M$, then $Q \cap N \in S$ and $Q \cap N \subseteq N$. Thus $N = Q \cap N \subseteq Q$ by minimality of $N$ in $S$. It follows that $N = P(M)$. Thus $P(M)$ is a finite intersection of prime $M$-ideals and it follows that $P(M)$ is a finite intersection of primitive $M$-ideals. So there exist primitive $M$-ideals $P_1, \ldots, P_n$ of $M$ such that $P(M) = \bigcap_{i=1}^{n} P_i$. Since $P_i$ is an $M$-ideal for every $1 \leq i \leq n$, $P_i \cdot M = P_i$ and so $P(M) = \bigcap_{i=1}^{n} P_i \cdot M = \bigcap_{i=1}^{n} P_i$. □

**Corollary 5.8.** Assume that $M$ is projective in $\sigma[M]$. If $M$ be an Artinian $M$-prime module, then $P(M) = (0)$. 

Proof. By Proposition 5.3, \((0)\) is a primitive \(M\)-ideal of \(M\). It follows that \(P(M) = (0)\) by Theorem 5.7.

Minimal \(M\)-prime submodules are defined in a natural way. By Zorn’s Lemma one can easily see that each \(M\)-prime submodule of a module \(X\) contains a minimal \(M\)-prime submodule of \(X\). In [18, Theorem 5.2], it is shown that every Noetherian module contain only finitely many minimal prime submodules. It is easy to show that if \(X\) is a Noetherian module, then \(X\) contain only finitely many minimal \(M\)-prime submodules.

We conclude this paper with the following interesting result, which is a generalization of [2, Theorem 2.1].

**Theorem 5.9.** Let \(X\) be a Noetherian \(R\)-module. If every \(M\)-prime submodule of \(X\) is virtually maximal, then \(X/\text{rad}_M(X)\) is an Artinian \(R\)-module.

**Proof.** By our hypotheses, for each \(M\)-prime submodule \(P\) of \(X\), \(X/P\) is a homogeneous semisimple \(R\)-module. Since \(X\) is a Noetherian \(R\)-module, \(X/P\) is also Noetherian. This implies that \(X/P\) is an Artinian \(R\)-module. On the other hand \(\text{rad}_M(X) = P_1 \cap \cdots \cap P_n\) where \(P_1, \ldots, P_n\) are all minimal \(M\)-prime submodules of \(M\). Thus \(X/P_1 \oplus \cdots \oplus X/P_n\) is also an Artinian \(R\)-module. It follows that \(X/\text{rad}_M(X)\) is an Artinian \(R\)-module. \(\square\)

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