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A CHARACTERIZATION OF PRIME IDEALS

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WE SHALL assume that all rings under consideration are associative rings with identity, and that all modules are unital. We show that a ring R is a prime ring if and only if every nonzero torsionless left R-module is faithful. This result can be extended to characterize the prime ideals of a ring R in terms of its left R-modules.

With each left R-module ${}_RM$ is associated an ideal $\mathrm{Ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$, the annihilator of M. The module ${}_RM$ is called faithful if $\mathrm{Ann}(M) = (0)$.

We may also define a left ideal $tr(M)$ associated with ${}_RM$, the sum of the left ideals $f(m)$, for all R-homomorphisms $f \in \mathrm{Hom}_R(M, R)$. For each $f \in \mathrm{Hom}_R(M, R)$ and each $r \in R$, the function g defined by $g(m) = f(m)r$, for all $m \in M$, is also a member of $\mathrm{Hom}_R(M, R)$. This can be used to show that in fact $tr(M)$ is an ideal of R.

The module ${}_RM$ is said to be torsionless if for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(m) \neq 0$. It is clear that each left ideal of R is torsionless when considered as a left R-module.

LEMMA. *Let ${}_RM$ be torsionless. Then M is faithful* $\Leftrightarrow \mathrm{Ann}(tr(M)) = (0)$.

PROOF. $\Rightarrow$. Suppose that ${}_RM$ is torsionless and faithful. Then $\mathrm{Ann}(M) = (0)$, and for each $0 \neq r \in R$ there exists $m \in M$ such that $rm \neq 0$. Since M is torsionless there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(rm) \neq 0$, and thus $f(m) \in tr(M)$ and $rf(m) \neq 0$. This shows that $\mathrm{Ann}(tr(M)) = (0)$.

$\Leftarrow$. If $r \in \mathrm{Ann}(M)$, then for each $m \in M$ and $f \in \mathrm{Hom}_R(M, R)$ we must have $rf(m) = f(rm) = 0$. Thus $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(tr(M))$, so $\mathrm{Ann}(tr(M)) = (0)$ implies $\mathrm{Ann}(M) = (0)$. Q.E.D.

A ring R is called a prime ring if for all ideals A, B of R, $A \cdot B = (0)$ implies $A = (0)$ or $B = (0)$. An ideal A of R, $A \neq R$, is called a prime ideal if the quotient ring R/A is a prime ring.

THEOREM 1. *The ring R is a prime ring $\Leftrightarrow$ every non-zero torsionless left R-module is faithful.*

PROOF. $\Rightarrow$. Assume that R is a prime ring and let ${}_RM$ be a non-zero torsionless left R-module. This implies that $tr(M) \neq 0$, and then $(\mathrm{Ann}(tr(M))) \cdot (tr(M)) = (0)$. Because we have assumed that R is a prime ring, we must have $\mathrm{Ann}(tr(M)) = (0)$, and it follows from the previous lemma that ${}_RM$ is faithful.

$\Leftarrow$. Assume that every non-zero torsionless left R-module is faithful. If A, B are ideals of R and $B \neq (0)$, then ${}_RB$ is a non-zero torsionless left R-module. By assumption, ${}_RB$ is faithful so $A \cdot B = (0)$ implies $A \subseteq \mathrm{Ann}(B) = (0)$. Thus for ideals A, B of R, $A \cdot B = (0)$ implies $A = (0)$ or $B = (0)$, and R is a prime ring. Q.E.D.

In order to generalize this result, it is convenient to adopt the following terminology.

DEFINITION. Let ${}_RM$ and ${}_RN$ be non-zero left R-modules. If for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, N)$ such that $f(m) \neq 0$, then we will write ${}_RM > {}_RN$. If ${}_RM > {}_RN$ and ${}_RN > {}_RM$, then we will write ${}_RM \sim {}_RN$.

With this definition we note that ${}_RM$ is a non-zero torsionless left R-module $\Leftrightarrow {}_RM > {}_RM$. If ${}_RM$ is any module, then for each $m \in M$ there exists $f \in \mathrm{Hom}_R(R, M)$ defined by $f(r) = rm$, for all $r \in R$, and since R has an identity, all R-homomorphisms from R to M are of this form. It is clear that ${}_R(R/\mathrm{Ann}(M)) > {}_RM$, and that ${}_RR > {}_RM \Leftrightarrow {}_RM$ is faithful. Thus Theorem 1 can be restated in the following form: The ring R is a prime ring if and only if ${}_RM > {}_RR \Rightarrow {}_RM \sim {}_RR$.

Suppose that ${}_RM > {}_RN$. If $r \notin \mathrm{Ann}(M)$, then there exists $m \in M$ such that $rm \neq 0$, so there exists $f \in \mathrm{Hom}_R(M, N)$ such that $f(rm) \neq 0$. Thus $rf(m) \neq 0$, which shows that $r \notin \mathrm{Ann}(N)$, and so ${}_RM > {}_RN$ implies $\mathrm{Ann}(N) \subseteq \mathrm{Ann}(M)$.

If ${}_RM > {}_RN$ and ${}_RN > {}_RP$, then for $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, N)$ such that $f(m) \neq 0$. Since ${}_RN > {}_RP$, there exists

$g \in \mathrm{Hom}_R(N, P)$ such that $g(f(m)) \neq 0$, and thus ${}_RM > {}_RP$. This also shows that ${}_RM \sim {}_RN$, ${}_RN \sim {}_RP \Rightarrow {}_RM \sim {}_RP$.

THEOREM 2. *Let A be an ideal of R, $A \neq R$. Then the following are equivalent:*

(i) *A is a prime ideal*;

(ii) ${}_RM > {}_R(R/A) \Rightarrow {}_RM \sim {}_R(R/A)$.

PROOF. If ${}_RM > {}_R(R/A)$, then $A = \mathrm{Ann}(R/A) \subseteq \mathrm{Ann}(M)$, and M is a left R/A-module. Every R-homomorphism from M to R/A is also an R/A-homomorphism, so ${}_{R/A}M > {}_{R/A}(R/A)$. Conversely, every left R/A-module can be regarded as a left R-module, and ${}_{R/A}M > {}_{R/A}(R/A) \Rightarrow {}_RM \sim {}_R(R/A)$. The theorem then follows immediately from the restatement of Theorem 1, since, by definition, A is a prime ideal if and only if R/A is a prime ring. Q.E.D.

THEOREM 3. *For a module ${}_RP$ the following are equivalent:*

(i) ${}_RM > {}_RP \Rightarrow {}_RM \sim {}_RP$;

(ii) ${}_RP \sim {}_R(R/A)$ *for a prime ideal A of R.*

PROOF. (i) $\Rightarrow$ (ii). Assume that ${}_RM > {}_RP \Rightarrow {}_RM \sim {}_RP$. Let $A = \mathrm{Ann}(P)$. Then $A \neq R$, and ${}_R(R/A) > {}_RP \Rightarrow {}_R(R/A) \sim {}_RP$. We will show that A is a prime ideal, using Theorem 2. If ${}_RM > {}_R(R/A)$, then ${}_RM > {}_RP$ and therefore ${}_RM \sim {}_RP$, from which it follows that ${}_RM \sim {}_R(R/A)$.

(ii) $\Rightarrow$ (i). Assume that ${}_RP \sim {}_R(R/A)$. Then ${}_RM > {}_RP$ implies ${}_RM > {}_R(R/A)$. By Theorem 2, ${}_RM \sim {}_R(R/A)$, and then ${}_RM \sim {}_RP$. Q.E.D.

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Comments

The results in this paper were put into the proper context in my paper *On Maximal Torsion Radicals*, Can. J. Math. **25** (1973), 712–726. The relevant part of that paper is included below.

Maximal radicals and prime ideals

A subfunctor ρ of the identity on R–Mod is a functor such that for all $M \in R$–Mod, $\rho(M)$ is a submodule of M, and if $f \in \mathrm{Hom}_R(M,N)$, then $f(\rho(M)) \subseteq \rho(N)$. Such a functor ρ is called a *radical* of R–Mod if $\rho(M/\rho(M)) = 0$ for all $M \in R$–Mod. A radical is *proper* if it is not the identity functor on R–Mod, or, equivalently, if $\rho(R) \neq R$.

If ρ and σ are radicals with $\rho(M) \subseteq \sigma(M)$ for all $M \in R$–Mod, we write $\rho \leq \sigma$, and if ρ is a radical then we call ρ a *maximal radical* if ρ is proper and for any other radical σ with $\rho \leq \sigma$, either $\rho = \sigma$ or σ is the identity on R–Mod.

If ρ is a radical, then a module ${}_RM$ is called ρ-*torsion* if $\rho(M) = M$, and ρ-*torsionfree* if $\rho(M) = M$. A submodule $M_0 \subseteq M$ is called ρ-*dense* if M/M_0 is ρ-torsion, and ρ-*closed* if M/M_0 is ρ-torsionfree. A (left) ideal A of R is a *maximal* ρ-closed (left) ideal if it is maximal in the set of proper ρ-closed (left) ideals.

For any module ${}_RN$, we define $\mathrm{rad}_N : R\text{–Mod} \to R\text{–Mod}$ by setting

$$\mathrm{rad}_N(M) = \cap_{f \in \mathrm{Hom}_R(M,N)} \ker(f)$$

for all $M \in R$–Mod. Then it can be shown that rad_N is a radical, and that $\mathrm{rad}_N(R) = \mathrm{Ann}(N)$.

If ρ is a radical and ${}_RN$ is ρ-torsionfree, then for any module ${}_RM$ and any $f \in \mathrm{Hom}_R(M,N)$ we must have $f(\rho(M)) \subseteq \rho(N) = 0$, and thus $\rho \leq \mathrm{rad}_N$. On the other hand, if $\rho \leq \mathrm{rad}_N$, then $\mathrm{rad}_N(N) = 0$ implies $\rho(N) = 0$. Therefore $\rho \leq \mathrm{rad}_N$ if and only if $\rho(N) = 0$. This result will prove to be useful in the characterization of maximal radicals.

LEMMA. *Let A be an ideal of R. Then A is a prime ideal if and only if* $\mathrm{rad}_N \geq \mathrm{rad}_{R/A}$ *implies* $\mathrm{Ann}(N) = A$*, for all nonzero* $N \in R$–Mod.

PROOF. Assume that A is a prime ideal and that $\mathrm{rad}_N \geq \mathrm{rad}_{R/A}$. Then $\mathrm{rad}_{R/A}(N) = 0$, so there exists $0 \neq f \in \mathrm{Hom}_R(N, R/A)$ since $N \neq 0$, which implies that $f(N) \neq 0$. Because R/A is a prime ring and $\mathrm{Ann}(N) \cdot f(N) = 0$, it follows that $\mathrm{Ann}(N) \subset A$. On the other hand, by assumption $\mathrm{Ann}(N) = \mathrm{rad}_N(R) \geq \mathrm{rad}_{R/A}(R) = \mathrm{Ann}(R/A) = A$.

Conversely, let B and C be ideals of R with $BC \subseteq A$. If $A \subset C$, then $C/A \neq 0$ and $\mathrm{rad}_{C/A} \geq \mathrm{rad}_{R/A}$. By assumption $A = \mathrm{Ann}(C/A) \supset B$, and this is sufficient to show that A is a prime ideal. Q.E.D.

Proposition. *Let A be an ideal of R. Then A is a prime ideal if and only if A is a maximal ρ-closed ideal for a radical ρ.*

Proof. If A is a prime ideal, let $\rho = \mathrm{rad}_{R/A}$. Then A is ρ-closed, and if B is any proper ρ-closed ideal, then $\mathrm{rad}_{R/A} \geq \mathrm{rad}_{R/A}$, and the lemma implies that $B = \mathrm{Ann}(R/B) \subseteq A$.

Conversely, if A is a maximal ρ-closed ideal and $\mathrm{rad}_N \geq \mathrm{rad}_{R/A}$ for some $0 \neq N \in R$–Mod, then $\mathrm{Ann}(N) \supseteq A$ and $\mathrm{Ann}(N)$ is ρ-closed since $\mathrm{rad}_N \geq \rho$. By assumption we must have $\mathrm{Ann}(N) = A$, and then the lemma implies that A is a prime ideal. Q.E.D.

Theorem. *Let ρ be a radical of R–Mod, with $\rho(R) = A$. Then ρ is a maximal radical if and only if $\rho = \mathrm{rad}_{R/A}$ and A is a prime ideal.*

Proof. Suppose that ρ is a maximal radical. Then $A \neq R$ and $\rho(R/A) = 0$ implies $\rho \leq \mathrm{rad}_{R/A}$, so $\rho = \mathrm{rad}_{R/A}$ since ρ is maximal. Furthermore, A is a maximal ρ-closed ideal since any larger ρ-closed ideal would determine a larger radical. The proposition then shows that A is a prime ideal.

Conversely, if A is a prime ideal and $\rho = \mathrm{rad}_{R/A}$, then the proof of the proposition shows that A is a maximal ρ-closed ideal. If α is a radical with $\alpha \geq \rho$, then $\alpha(R)$ is ρ-closed and contains A. Hence either $\alpha(R) = R$ or $\alpha(R) = A$ and $\alpha \leq \mathrm{rad}_{R/A} = \rho$. Q.E.D.

Corollary. *Every proper radical of R–Mod is contained in a maximal radical if and only if for each $0 \neq M \in R$–Mod there exists a submodule $M_0 \subseteq M$ such that $\mathrm{Ann}(M_0)$ is a prime ideal.*

Proof. If $0 \neq M \in R$–Mod and every proper radical is contained in a maximal radical, then the theorem shows that there exists a prime ideal P with $\mathrm{rad}_{R/P} \geq \mathrm{rad}_M$. Let $M_0 = \{m \in M : Pm = 0\}$. It can easily be shown that $\mathrm{Ann}(M_0) = P$.

Conversely, let ρ be a proper radical with $\rho(R) = A$. By assumption there exists a left ideal $A \supset B$ for which $\mathrm{Ann}(B/A) = P$ is a prime ideal. Thus $\rho \leq \mathrm{rad}_{R/A} \leq \mathrm{rad}_{R/P}$, and $\mathrm{rad}_{R/P}$ is a maximal radical. Q.E.D.