Recently there has been considerable interest in extending to non-commutative rings the techniques of localization which have proved to be useful in the study of commutative rings. The notion of a quotient category, as utilized by Gabriel [5], has played a fundamental role in one approach to the problem. The constructions considered by Gabriel include quotient categories determined either by localizing Serre subcategories, certain filters of left ideals, or injective modules. (A special case of the latter method had been introduced by Findlay and Lambek [4].) Further equivalent notions are the torsion radicals of Maranda [12] (called idempotent kernel functors by Goldman [6]) and the (hereditary) torsion theories of Dickson [3] and Lambek [8]. The reader is also referred to the Walkers’ paper [20] and the recent expositions of Lambek [9,11], Morita [14], and Stenström [18], which contain extensive bibliographies.

A major difficulty in this approach is that the quotient category in general need not coincide with the category of modules over the corresponding ring of quotients. This paper gives an expository account
of the conditions under which the two categories coincide. It is intended to give the non-expert quick access to some of the basic definitions and theorems in noncommutative localization, and an exposure to some of the techniques of the area. It assumes a knowledge of elementary ring theory and category theory.

It seems appropriate to first review the relevant properties of localization in commutative rings. Let \( R \) be a commutative ring (all rings will be associative rings with identity element, and all modules will be unital), and let \( S \) be a multiplicative system in \( R \) (\( S \) is a multiplicatively closed subset of \( R \) with \( 0 \notin S \)). The ring of quotients determined by \( S \) is a ring \( R_S \) and ring homomorphism \( \alpha : R \to R_S \) with \( \alpha(s) \) invertible for all \( s \in S \), such that for each ring homomorphism \( \eta : R \to R' \) with \( \eta(s) \) invertible for all \( s \in S \), there exists a unique homomorphism \( \theta : R_S \to R' \) with \( \eta = \theta \alpha \).

For a module \( _RM \) the construction of a module of quotients \( M_S \) over \( R_S \) proceeds as follows. The first step is to factor out the submodule \( \text{rad}_S(M) = \{ m \in M \mid sm = 0 \text{ for some } s \in S \} \), since no element of \( M_S \) can be annihilated by an invertible element of \( R_S \). Note that \( \text{rad}_S(M/\text{rad}_S(M)) = 0 \). This construction defines a left exact subfunctor \( \text{rad}_S \) of the identity on the category of \( R \)-modules \( R\text{-Mod} \), and motivates the general definition of a torsion radical, since it actually determines the module of quotients completely.

The module of quotients \( M_S \) is given by the set of pairs \( m/s \), with \( m \in M/\text{rad}_S(M) \) and \( s \in S \), with \( m_1/s_1 = m_2/s_2 \) if \( s_2m_1 = s_1m_2 \). If \( \text{rad}_S(M) = 0 \), then for any element \( 0 \neq m/s \in M_S \), \( 0 \neq s(m/s) \in M \), so that each nonzero submodule of \( M_S \) has nonzero intersection with \( M \), and thus \( M_S \) is an essential extension of \( M \). This shows that if \( \text{rad}_S(M) = 0 \), then \( M_S \) can be viewed as a submodule of the injective envelope \( E(M) \) of \( M \) (which is a maximal essential extension of \( M \)).
in fact, $M_S = \{x \in E(M) \mid sx \in M \text{ for some } s \in S\}$, so that $M_S/M = \text{rad}_S(E(M)/M)$. In general, then, for a torsion radical $\sigma$, the module of quotients $M_\sigma$ is constructed as the inverse image in $E(M'')$ of $\text{rad}_\sigma(E(M'')/M'')$, where $M'' = M/\text{rad}_\sigma(M)$.

The module of quotients $M_S$ has the property that for each nonzero element $x \in M_S$ and each $s \in S$ there exists a unique element $y = x/s$ such that $sy = x$. In terms of homomorphisms, this states that for each $s \in S$, any $R$-homomorphism $f : Rs \to M_S$ can be extended uniquely to $g : R \to M_S$. For a torsion radical $\sigma$, this motivates the notion of a $\sigma$-torsionfree, $\sigma$-injective module, and the full subcategory $R\text{-Mod}/\sigma$ of $\sigma$-torsionfree, $\sigma$-injective $R$-modules is called the quotient category determined by $\sigma$.

The quotient functor $Q_\sigma : R\text{-Mod} \to R\text{-Mod}/\sigma$ is defined on modules by $Q_\sigma(M) = M_\sigma$. For a multiplicative system $S$ in a commutative ring, $M_S$ is naturally isomorphic to $R_S \otimes_R M$, and $R_S\text{-Mod}$ is equivalent to the quotient category $R\text{-Mod}/S$. Since the quotient functor is exact, $R_S$ must be flat as an $R$-module, and since every element of $R_S$ has the form $\alpha(r)/s$ for some $r \in R$ and $s \in S$, it can be shown that if $\beta : R_S \to R'$, $\gamma : R_S \to R'$ are ring homomorphisms with $\beta \alpha = \gamma \alpha$, then $\beta = \gamma$, so that $\alpha$ is an epimorphism in the category of rings. In general, the quotient functor $Q_\sigma$ is naturally isomorphic to the functor $R_\sigma \otimes_R -$ if and only if the inclusion functor $R\text{-Mod}/\sigma \to R_\sigma\text{-Mod}$ is an equivalence, and in this case $Q_\sigma$ is said to be a perfect quotient functor. Furthermore, there is a one-to-one correspondence between perfect quotient functors and flat epimorphic images of $R$. These localizations were studied by Silver [17]. If $R$ is a Dedekind domain with field of quotients $K$, then every subring of $K$ which contains $F$ is a flat epimorphic extension of $R$, and so all such subrings are rings of quotients with perfect quotient functors. But every ring between $R$ and $K$ is a ring of quotients.
of $R$ with respect to a multiplicative system if and only if the group of ideal classes of $R$ is a torsion group ([6]), which shows that it is natural to study perfect quotient functors even for commutative rings.

Various properties of torsion radicals, quotient categories, and rings of quotients are summarized in Section 1. These give the necessary background for Section 2, in which several classes of objects in the quotient category are studied. The final section deals with perfect quotient functors. Some of the techniques utilized appear to be novel in that they make heavy use of the quotient category. Most of the proofs have been omitted in the first section, and have only been sketched in the last two sections.

§1. Some background results

A functor $\sigma : R-\text{Mod} \to R-\text{Mod}$ is called a torsion radical if for all modules $R\mathcal{M}$, $R\mathcal{N}$ and all $f \in \text{Hom}_R(M, N)$, $\sigma M \subseteq M$, $f(\sigma M) \subseteq \sigma N$, $\sigma(M/\sigma M) = 0$ and $\sigma M' = \sigma M \cap M'$ for all submodules $M' \subseteq M$. The submodule $\sigma(M)$ will be denoted by $\text{rad}_\sigma(M)$. A module $R\mathcal{M}$ is called $\sigma$-torsion if $\text{rad}_\sigma(M) = M$, and $\sigma$-torsionfree if $\text{rad}_\sigma(M) = 0$. Note that $\text{rad}_\sigma(M)$ can be characterized either as the sum of $\sigma$-torsion submodules of $M$ or as the intersection of submodules of $M$ whose factor is $\sigma$-torsionfree.

The class of $\sigma$-torsion modules is closed under formation of epimorphic images, direct sums, (group) extensions, and submodules, and moreover, each such class is the torsion class of some torsion radical. Similarly, the class of $\sigma$-torsionfree modules is closed under submodules, direct products, (group) extensions, and essential extensions, and each such class is the torsionfree class of some torsion radical. Furthermore, $R\mathcal{M}$ is $\sigma$-torsion iff $\text{Hom}_R(M, N) = 0$ for all $\sigma$-torsionfree modules $R\mathcal{N}$, and $N$ is $\sigma$-torsionfree iff $\text{Hom}_R(M, N) = 0$ for all $\sigma$-torsion modules $M$. 406
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If \( \sigma, \tau \) are torsion radicals such that \( \text{rad} \sigma(M) \subseteq \text{rad} \tau(M) \) for all modules \( _RM \), then the notation \( \sigma \leq \tau \) is used, and \( \tau \) is said to be larger than \( \sigma \). If \( W \in R\text{-Mod} \), then for any module \( _RM \) let \( \text{rad}_W(M) \) be the intersection of all kernels of homomorphisms from \( M \) into \( W \). The last statement of the following theorem is due to Jans [7], where the injective envelope of the direct product of a representative set of cyclic \( \sigma \)-torsionfree \( R \)-modules can be taken as \( W \).

**THEOREM (1.1).** If \( RW \) is injective, then \( \text{rad}_W \) defines a torsion radical of \( R\text{-Mod} \), and \( \sigma = \text{rad}_E(M) \) is the largest torsion radical for which \( _RM \) is \( \sigma \)-torsionfree. If \( \sigma \) is any torsion radical of \( R\text{-Mod} \), then there exists an injective module \( RW \) such that \( \sigma = \text{rad}_W \).

A torsion radical \( \sigma \) defines a closure operation on submodules \( M' \subseteq M \) by letting \( C_\sigma(M') \) be the inverse image in \( M \) of \( \text{rad}_\sigma(M/M') \). The submodule \( M' \) is called \( \sigma \)-dense if \( C_\sigma(M') = M \), that is, if \( M/M' \) is \( \sigma \)-torsion, and \( \sigma \)-closed if \( C_\sigma(M') = M' \), which occurs iff \( M/M' \) is \( \sigma \)-torsionfree. The set of \( \sigma \)-dense left ideals of \( R \) will be denoted by \( D_\sigma \). The closure of \( M' \) can be described as

\[
C_\sigma(M') = \{ m \in M \mid Dm \subseteq M' \text{ for some } D \in D_\sigma \},
\]

or if \( \sigma = \text{rad}_W \), as

\[
\{ m \in M \mid f(m) = 0 \text{ for all } f \in \text{Hom}_R(M,N) \text{ such that } f(M') = 0 \}.
\]

It follows immediately that if \( f \in \text{Hom}_R(M,N) \) and \( f(M') \subseteq N' \), then \( f(C_\sigma(M')) \subseteq C_\sigma(N') \), and so if \( N \) is \( \sigma \)-torsionfree and \( f, g \in \text{Hom}_R(M,N) \) agree on \( M' \), then they must agree on \( C_\sigma(M') \). For convenience, the \( \sigma \)-closure of \( M \) in its injective envelope \( E(M) \) will be denoted by \( E_\sigma(M) \).
The σ-dense submodules satisfy the following properties. If \( M_1 \) and \( M_2 \) are submodules of \( M \), then (i) if \( M_1 \subseteq M_2 \) and \( M_1 \) is σ-dense, then \( M_2 \) is σ-dense; (ii) if \( M_1 \) and \( M_2 \) are σ-dense, then so is \( M_1 \cap M_2 \); (iii) \( M_1 \) is σ-dense iff \( \{ r \in R \mid rm \in M_1 \} \in \mathcal{D}_\sigma \), for all \( m \in M \); (iv) if \( M_1 \subseteq M_2 \) with \( M_1 \) σ-dense in \( M_2 \) and \( M_2 \) σ-dense in \( M \), then \( M_1 \) is σ-dense in \( M \). A nonempty set \( \mathcal{D} \) of left ideals of \( R \) is called an idempotent topologizing filter (IT-filter) if the following conditions are satisfied: (i) \( \mathcal{D} \in \mathcal{D} \) and \( \mathcal{D} \subseteq A \subseteq R \) implies \( A \in \mathcal{D} \); (ii) \( D_1, D_2 \in \mathcal{D} \) implies \( D_1 \cap D_2 \in \mathcal{D} \); (iii) \( D \in \mathcal{D} \) implies \( Dr^{-1} \in \mathcal{D} \) for all \( r \in R \), where \( Dr^{-1} = \{ x \in R \mid xr \in D \} \); (iv) \( D \in \mathcal{D} \), \( A \subseteq D \), and \( Ad^{-1} \in \mathcal{D} \) for all \( d \in D \) implies \( A \in \mathcal{D} \). The preceding remarks on σ-dense submodules make it clear that \( \mathcal{D}_\sigma \) is an IT-filter. Conversely, if \( \mathcal{D} \) is an IT-filter, then letting \( \text{rad}_D(M) = \{ m \in M \mid \text{Ann}(m) \in \mathcal{D} \} \) defines a torsion radical. This can be used to establish the following theorem of Maranda [12], also implicit in Gabriel [5].

**THEOREM (1.2).** There is a one-to-one correspondence between torsion radicals of \( R \)-Mod and IT-filters of left ideals of \( R \).

A module \( R \mathcal{M} \) is called σ-injective if each homomorphism \( f : N' \to M \) such that \( N' \) is a σ-dense submodule of \( R \mathcal{N} \) can be extended to \( N \). This reduces to the usual definition of injectivity if \( \sigma \) is the identity functor. It can be shown in the standard way that a direct product of modules if σ-injective iff each factor is σ-injective. It is almost clear that \( M \) is σ-injective iff \( M = E_\sigma(M) \), and in fact \( E_\sigma(M) \) defines a “σ-injective envelope” of \( M \). Baer’s condition holds in this situation, in that \( M \) is σ-injective iff each homomorphism \( f : D \to M \) such that \( D \in \mathcal{D}_\sigma \) can be extended to \( R \).
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PROPOSITION (1.3). The following conditions are equivalent for $M \in R$–$\text{Mod}$:

(a). $M$ is $\sigma$-torsionfree and $\sigma$-injective.

(b). Each homomorphism $f : N' \to M$ such that $N'$ is a $\sigma$-dense submodule of $R N$ can be extended uniquely to $N$.

(c). $\text{Hom}_R(N, M) = 0 = \text{Ext}_1^1(N, M)$ for all $\sigma$-torsion modules $N$.

(d). $E(M)$ and $E(M)/M$ are $\sigma$-torsionfree.

The full subcategory of $R$–$\text{Mod}$ determined by all $\sigma$-torsionfree, $\sigma$-injective modules will be denoted by $R$–$\text{Mod}/\sigma$. If $f \in \text{Hom}_R(M, N)$ and $M, N \in R$–$\text{Mod}/\sigma$, then $\text{ker}(f)$ is $\sigma$-closed in $M$. Since any submodule of a $\sigma$-torsionfree module is $\sigma$-torsionfree, and a $\sigma$-closed submodule of a $\sigma$-injective module is $\sigma$-injective, $\text{ker}(f) \in R$–$\text{Mod}/\sigma$. On the other hand if $f$ is a monomorphism, then $N/f(M)$ is $\sigma$-torsionfree, since a $\sigma$-injective submodule of a $\sigma$-torsionfree module is $\sigma$-closed, and so $M$ is the kernel of the natural homomorphism $N \to E_\sigma(N/f(M))$. This shows that the category $R$–$\text{Mod}/\sigma$ has kernels, and that every monomorphism is a kernel. Since it clearly has finite direct sums and a zero object, to show that $R$–$\text{Mod}/\sigma$ is an abelian category it is sufficient to show that it has cokernels and that every epimorphism is a cokernel.

It is convenient to define the quotient functor $Q_\sigma$ at this point.

Let $Q_\sigma(M) = E_\sigma(M/\text{rad}_\sigma(M))$, for $M \in R$–$\text{Mod}$. Then if $R N \in R$–$\text{Mod}/\sigma$ and $f : M \to N$, $f$ factors through $M/\text{rad}_\sigma(M)$ since $f(\text{rad}_\sigma(M)) \subseteq \text{rad}_\sigma(N) = 0$. Since $N$ is $\sigma$-torsionfree and $\sigma$-injective, by Proposition 1.3 there is a unique extension of $f$ to $E_\sigma(M/\text{rad}_\sigma(M))$ since $M/\text{rad}_\sigma(M)$ is a $\sigma$-dense submodule of $E_\sigma(M/\text{rad}_\sigma(M))$. This shows how to define the functor $Q_\sigma$ on homomorphisms, and more importantly shows that $Q_\sigma$ is a left adjoint of the inclusion $U_\sigma : R$–$\text{Mod}/\sigma \to R$–$\text{Mod}$.

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If \( f : M \to N \) is a homomorphism in \( R-\text{Mod}/\sigma \), then \( Q_\sigma(\text{coker}(f)) \) serves as the cokernel of \( f \) in \( R-\text{Mod}/\sigma \). Furthermore, if \( f \) is an epimorphism in \( R-\text{Mod}/\sigma \), then \( Q_\sigma(\text{coker}(f)) = 0 \) and so \( f(M) \) must be \( \sigma \)-dense in \( N \), which shows that \( N \) is the cokernel of the inclusion \( \ker(f) \to M \). Thus \( R-\text{Mod}/\sigma \) is an abelian category, and so \( Q_\sigma \) is right exact since it is a left adjoint. (It is easy to show that \( Q_\sigma \) preserves monomorphisms, so in fact \( Q_\sigma \) is exact. It is important to note that the inclusion functor \( U_\sigma \) need not be exact, since an epimorphism in \( R-\text{Mod}/\sigma \) may not be an epimorphism in \( R-\text{Mod} \), and in fact \( U_\sigma Q_\sigma \) need not be exact.) The above remarks can be used to establish the following important theorem. (See Gabriel [5] or Mitchell [13].)

**THEOREM (1.4)** Let \( \sigma \) be a torsion radical of \( R-\text{Mod} \). Then the quotient category \( R-\text{Mod}/\sigma \) is an abelian category, and the inclusion function \( U_\sigma : R-\text{Mod}/\sigma \to R-\text{Mod} \) has an exact left adjoint \( Q_\sigma : R-\text{Mod} \to R-\text{Mod}/\sigma \).

As shown by Gabriel [5], an alternate method of constructing \( Q_\sigma(M) \) is to take the direct limit of \( \text{Hom}_R(D, M/\text{rad}_\sigma(M)) \) over all \( D \in D_\sigma \). Theorem 1.4 has a converse, in that any full abelian subcategory of \( R-\text{Mod} \) whose inclusion functor has an exact left adjoint is equivalent to a subcategory of the form \( R-\text{Mod}/\sigma \) for some torsion radical \( \sigma \). Another useful characterization of subcategories of this form is given by a proposition due to Morita [14].

**PROPOSITION (1.5)** Let \( \mathcal{A} \) be a full subcategory of \( R-\text{Mod} \). Then \( \mathcal{A} = R-\text{Mod}/\sigma \) for some torsion radical \( \sigma \) \iff \( \mathcal{A} \) is closed under isomorphisms, direct summands, direct products, and \( M \in \mathcal{A} \) iff \( \text{E}(M), \text{E}(\text{E}(M)/M) \in \mathcal{A} \).
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The module of quotients \( Q_\sigma(M) \) will be denoted simply by \( M_\sigma \). For any module \( M \in R{\text{-}}Mod/\sigma \), and any element \( m \in M \), the homomorphism \([r \mapsto rm] : R \to M\) defined by multiplication can be extended uniquely to \( \rho_m : R_\sigma \to M \). For any element \( q \in R_\sigma \), \( \rho_q \) can be used to define right multiplication by \( q \), and this induces a ring structure on \( R_\sigma \). Furthermore, any module \( M \in R{\text{-}}Mod/\sigma \) becomes a left \( R_\sigma \)-module by defining \( qm = \rho_m(q) \), for all \( q \in R \) and \( m \in M \). In fact, a \( \sigma \)-torsionfree \( R \)-module can be given an \( R_\sigma \)-module structure iff it is generated in the categorical sense by \( RR_\sigma \) ([2]). The ring \( R_\sigma \) is called the ring of quotients determined by \( \sigma \).

THEOREM (1.6). The module \( R_\sigma \) has a ring structure, and every \( R \)-module in \( R{\text{-}}Mod/\sigma \) is an \( R_\sigma \)-module. Every \( R \)-homomorphism in \( R{\text{-}}Mod/\sigma \) is an \( R_\sigma \)-homomorphism.

A module \( M \in R{\text{-}}Mod/\sigma \) is injective as an \( R_\sigma \)-module iff it is injective as an \( R \)-module, and so it follows from Proposition 1.5 that \( R{\text{-}}Mod/\sigma \) is a quotient category of \( R_\sigma{\text{-}}Mod \) as well as of \( R{\text{-}}Mod \). It can be shown that there exists an injective module \( rW \) with \( \sigma = \text{rad}_W \) such that \( R/\text{Ann}(W) \) can be embedded in \( W \), and that \( R_\sigma \) is isomorphic to the bicommutator (double centralizer) of \( W \). More generally, note that \( R_\sigma \) is isomorphic to the bicommutator of \( W \) provided only that \( W \) is finitely generated over its endomorphism ring, so this gives another method of constructing rings of quotients.

The ring of quotients \( Q_{\text{max}}(R) \) determined by \( \text{rad}_{E(R)} \) is called the complete ring of left quotients of \( R \). A torsion radical \( \sigma \) will be called complete if \( \sigma = \text{rad}_{E(R/K)} \) for \( K = \text{rad}_\sigma(R) \). In this case, since \( E(R/K) \) is annihilated by \( K \), it is the \( R/K \)-injective envelope of \( R/K \), and so \( R_\sigma = Q_{\text{max}}(R/K) \).
§2. Objects in the quotient category

In the remaining sections $\sigma$ will be a fixed torsion radical. The quotient category $R{-}\text{Mod}/\sigma$ has direct products and a generator, $R_\sigma$. The categorical direct sum $\bigsqcup_{\alpha \in I} M_\alpha$ of a set of modules $\{M_\alpha\}_{\alpha \in I}$ in $R{-}\text{Mod}/\sigma$ is given by $E_\sigma(\bigoplus_{\alpha \in I} M_\alpha)$. The next propositions investigate objects in $R{-}\text{Mod}/\sigma$ which are injective, projective, and have various chain conditions or finiteness conditions.

**PROPOSITION (2.1).** Let $M, W \in R{-}\text{Mod}$, with $R W$ injective.

(a). There is a one-to-one correspondence between subobjects of $M_\sigma$ in $R{-}\text{Mod}/\sigma$ and $\sigma$-closed submodules of $M$.

(b). If $M \in R{-}\text{Mod}/\sigma$, then $M$ is injective (quasi-injective) in $R{-}\text{Mod}/\sigma \iff M$ is injective (quasi-injective) in $R{-}\text{Mod}$.

(c). $W$ is an injective cogenerator in $R{-}\text{Mod}/\sigma \iff \text{rad} W = \sigma$.

Proof. (a). Since every $\sigma$-closed submodule of $M$ contains $\text{rad}_\sigma(M)$, $\sigma$-closed submodules of $M/\text{rad}_\sigma(M)$ correspond to those of $M$. If $M$ is $\sigma$-torsionfree, then there is a one-to-one correspondence between $\sigma$-closed submodules of $M$ and $E_\sigma(M)$, where a submodule is extended by taking its $\sigma$-closure in $E_\sigma(M)$ and contracted by intersection. Finally, the subobjects of a module in $R{-}\text{Mod}/\sigma$ are just its $\sigma$-closed submodules.

(b). The inclusion functor $U_\sigma$ preserves injectives since it has an exact left adjoint, and if $R M$ is injective, then it is obviously injective in $R{-}\text{Mod}/\sigma$. This shows that injective envelopes are the same in both categories, and so the statement on quasi-injectives follows from the characterization of quasi-injective objects as fully invariant subobjects of their injective envelopes.
(c). W is a cogenerator in $R\text{-Mod}/\sigma$ iff each module in $R\text{-Mod}/\sigma$ can be embedded in a direct product of copies of W, and this occurs iff $\text{rad}_W = \sigma$.

**PROPOSITION (2.2).** If $\sigma = \text{rad}_{E(X)}$ for $X \in R\text{-Mod}$ and $\text{rad}_\sigma(R) = K$, then the following conditions are equivalent.

(a). Every object in $R\text{-Mod}/\sigma$ is injective.

(b). Every essential left ideal of $R/K$ is $\sigma$-dense in $R/K$.

(c). $X$ is a nonsingular $R/K$-module.

Proof. (a) $\Rightarrow$ (b). If A is an essential left ideal of $R/K$, then $E(R/K) = E(A)$, and if every object in $R\text{-Mod}/\sigma$ is injective, then $Q_\sigma(R/K) = Q_\sigma(A)$, so A is $\sigma$-dense in $R/K$.

(b) $\Rightarrow$ (c). Recall that $X$ is nonsingular if no nonzero element has an essential annihilator. Since $\sigma = \text{rad}_{E(X)}$, the annihilator of any element of $X$ is $\sigma$-closed in $R/K$, so by assumption it cannot be essential.

(c) $\Rightarrow$ (a). If $M \in R\text{-Mod}/\sigma$, then $E(M)$ and $E(M)/M$ are $\sigma$-torsionfree, and so they are $R/K$-modules. The annihilator in $R/K$ of any element of $E(M)/M$ is essential and $\sigma$-closed in $R/K$, and it can also be shown to be $\sigma$-dense since $\sigma = \text{rad}_{E(X)}$ and $X$ is nonsingular over $R/K$. This is a contradiction, so $M = E(M)$, and thus $M$ is injective in $R\text{-Mod}/\sigma$.

The above proposition is a slight modification of a result of Lambe-

bek [10]. Note that in this case $R_\sigma$ is von Neumann regular since every principal left ideal belongs to $R\text{-Mod}/\sigma$ and hence is a direct summand of $R_\sigma$.

By Proposition 2.1, for $RM$ the module of quotients $M_\sigma$ is Noether-

ian (Artinian) in $R\text{-Mod}/\sigma$ iff the $\sigma$-closed submodules of $M$ satisfy the ascending chain condition (d.c.c., respectively). A module $M \in R\text{-Mod}/\sigma$
will be called \( \textit{finitely generated} \) (in \( R\text{-}\text{Mod}/\sigma \)) if for any epimorphism 
\( \prod_{\alpha \in I} M_\alpha \to M \) in \( R\text{-}\text{Mod}/\sigma \) there is a finite subset \( F \subseteq I \) such that 
the natural morphism \( \prod_{\alpha \in F} M_\alpha \to \prod_{\alpha \in I} M_\alpha \to M \) is an epimorphism. The next proposition 
can be used to show that for \( _RM, M_\sigma \) is Noetherian iff every subobject is finitely generated (in \( R\text{-}\text{Mod}/\sigma \)).

\textbf{PROPOSITION (2.3).} Let \( M \in R\text{-}\text{Mod} \). Then \( M_\sigma \) is \textit{finitely generated} in \( R\text{-}\text{Mod}/\sigma \) \( \iff \) every \( \sigma \)-dense submodule of \( M \) contains an 
\( R \)-\textit{finitely generated} \( \sigma \)-dense submodule.

Proof. \( \Rightarrow \). If \( M_\sigma \) is finitely generated in \( R\text{-}\text{Mod}/\sigma \), and \( M' \) is a \( \sigma \)-dense submodule, then \( M' \) is a quotient of a free module \( R^I \). Applying 
\( Q_\sigma \) yields \( \prod_I Q_\sigma(R) \to Q_\sigma(M') = M_\sigma \), an epimorphism, since \( Q_\sigma \) is 
exact and preserves direct sums. Since \( M_\sigma \) is finitely generated, for some finite subset \( F \) the image of \( R^F \to M' \) is \( \sigma \)-dense in \( M' \).

\( \Leftarrow \). If the condition holds for \( M \), then it is inherited by \( M/\text{rad}_\sigma(M) \) 
and then by \( E_\sigma(M/\text{rad}_\sigma(M)) = M_\sigma \). The conclusion then follows from 
the fact that a module in \( R\text{-}\text{Mod}/\sigma \) is finitely generated there iff every 
\( \sigma \)-dense sum of \( \sigma \)-closed submodules has a finite subcollection whose 
sum is \( \sigma \)-dense.

\textbf{COROLLARY (2.4).} Let \( X \in R\text{-}\text{Mod}, \sigma = \text{rad}_{E(X)} \), and \( K = \text{rad}_\sigma(R) \). If \( X \) is nonsingular as an \( R/K \)-module, then \( X_\sigma \) is \textit{finitely generated} in \( R\text{-}\text{Mod}/\sigma \) \( \iff R \) \( X \) has \textit{finite uniform dimension}.

Proof. If \( X \) is a nonsingular \( R/K \)-module, then its \( \sigma \)-dense submodules 
coincide with its essential submodules, and so every essential submodule 
contains a finitely generated essential submodule iff \( _R X \) has finite 
uniform dimension (that is, \( X \) does not contain an infinite direct sum 
of nonzero submodules).
A module $M \in R\text{-}Mod/\sigma$ is called small if $\text{Hom}_R(M, -)$ preserves direct sums. This occurs iff each homomorphism $f : M \to \prod_{\alpha \in I} M_\alpha$ admits a factorization through $\prod_{\alpha \in I} M_\alpha$ for some finite subset $F \subseteq I$. This motivates the next definition. A module $M \in R\text{-}Mod$ will be called $\sigma$-small if for each homomorphism $f : M \to \bigoplus_{\alpha \in I} M_\alpha$ such that $M_\alpha \in R\text{-}Mod/\sigma$ for all $\alpha \in I$, $f$ can be factored through the direct sum of a finite subcollection. The next proposition follows rather quickly from the definition.

**Proposition (2.5).** Let $M \in R\text{-}Mod$. Then $M_\sigma$ is small in $R\text{-}Mod/\sigma \iff$ every $\sigma$-dense submodule of $M$ is $\sigma$-small.

**Proposition (2.6).** Let $D \in D_\sigma$. Then $R_D$ is $\sigma$-small $\iff$ for each ascending chain $\{A_i\}_{i=1}^\infty$ of left ideals with $\cup_{i=1}^\infty A_i = D$, $A_i \in D_\sigma$ for some $i$.

Proof. $\Rightarrow$. If $D \in D_\sigma$ and $D = \cup_{i=1}^\infty A_i$, then the natural homomorphism $D \to \bigoplus_{i=1}^\infty Q_\sigma(R/A_i)$ factors through a finite direct sum, and so $Q_\sigma(R/A_i) = 0$ for some $i$, which shows that $A_i$ is $\sigma$-dense.

$\Leftarrow$. Given a homomorphism $g : D \to \bigoplus_{\alpha \in I} M_\alpha$, extend $g$ to $f : R \to \prod_{\alpha \in I} M_\alpha$. Let $F$ be the set of indices $\alpha$ such that $f(1)_\alpha \neq 0$. For any countable subset of $F$ let $A_i = \{d \in D \mid f_n(d) = 0 \text{ for } n \geq i\}$. For any $d \in D$, $f_n(d) = 0$ for all but finitely many components, so $\cup_{i=1}^\infty A_i = D$, and therefore by assumption $A_i$ is $\sigma$-dense for some $i$. But now $A_i f_n(1) = 0$ for all $n \geq i$, and this implies $f_n(1) = 0$ for all $n \geq i$ since each $M_n$ is $\sigma$-torsionfree. This shows that every countable subset of $F$ must be finite, and hence $F$ must be finite, so that $g$ factors through the direct sum of a finite subcollection of modules $M_\alpha$, and $D$ is $\sigma$-small.
A module \( R \) will be called \( \sigma \)-projective if each homomorphism \( f : P \to N \) such that \( g : M \to N \) is an epimorphism, with \( M \in R-\text{Mod}/\sigma \) and \( N \) \( \sigma \)-torsionfree, can be lifted to \( h : P \to N \) with \( gh = f \). This is equivalent to the condition that each homomorphism \( f : P \to N \) such that \( g : M \to N \) is an epimorphism, with \( M, N \) \( \sigma \)-torsionfree, can be lifted to a \( \sigma \)-dense submodule \( P' \) of \( P \). (The latter is the definition used by Goldman [6].)

**Proposition (2.7).** Let \( P \in R-\text{Mod} \). Then \( P_\sigma \) is projective in \( R-\text{Mod}/\sigma \) \( \iff \) every \( \sigma \)-dense submodule of \( P \) is \( \sigma \)-projective.

**Proof.** \( \Rightarrow \). If \( P' \) is \( \sigma \)-dense in \( P \) and \( f : P' \to N \) with \( g : M \to N \) an epimorphism, then \( f \) extends uniquely to \( f_\sigma : Q_\sigma(P') \to Q_\sigma(N) \).

If \( M \in R-\text{Mod}/\sigma \) and \( N \) is \( \sigma \)-torsionfree, then \( M \to Q_\sigma(N) \) is an epimorphism in \( R-\text{Mod}/\sigma \), and so \( f_\sigma \) lifts to \( M \) because \( Q_\sigma(P') \) is isomorphic to \( P_\sigma \) and therefore projective in \( R-\text{Mod}/\sigma \).

\( \Leftarrow \). If \( g : M \to N \) is an epimorphism in \( R-\text{Mod}/\sigma \), and \( f : P_\sigma \to N \), then the inverse image \( P' \subseteq P \) of \( g(M) \) is \( \sigma \)-dense in \( P \) since \( g(M) \) is \( \sigma \)-dense in \( N \). By assumption the homomorphism \( P' \to g(M) \) can be lifted to \( M \), and then it can be extended uniquely to \( P_\sigma \), and the extension to \( P_\sigma \) is the required lifting of \( f \).

A module \( R \) is called **monoform** if every nonzero homomorphism from a submodule of \( M \) into \( M \) is a monomorphism. A torsion radical \( \sigma \) is said to be a **prime torsion radical** ([6]) if \( \sigma = \text{rad}_{E(M)} \) with \( M \) monoform. The next proposition gives several characterizations of monoform modules, and shows that a torsion radical \( \sigma \) is a prime torsion radical iff the quotient category \( R-\text{Mod}/\sigma \) has an injective cogenerator which is the injective envelope of a simple object.
PROPOSITION (2.8). Let $M \in R$–Mod, and let $\sigma = \text{rad}_{E(M)}$. Then the following conditions are equivalent:

(a). $M$ is monoform.

(b). $M_\sigma$ is simple in $R$–Mod/$\sigma$.

(c). $M_\sigma$ is quasi-injective in $R$–Mod, and $\text{End}_R(M_\sigma)$ is a division ring.

(d). $\text{End}_R(M) = \text{End}_R(M_\sigma)$ is a division ring, where $M$ is the quasi-injective envelope of $RM$.

Proof. (a)⇒(b). If $M$ is monoform, then $\text{Hom}_R(M_2/M_1, M) = 0$ for all submodules $0 \neq M_1 \subseteq M_2 \subseteq M_3$, or equivalently, $\text{Hom}_R(M/M_1, E(M)) = 0$ for all $0 \neq M_1 \subseteq M$. Thus every nonzero submodule of $M$ is $\sigma$-dense, and so $M_\sigma$ has no nontrivial subobjects in $R$–Mod/$\sigma$.

(b)⇒(c). If $M_\sigma$ is simple, then its endomorphism ring in $R$–Mod/$\sigma$, which is just $\text{End}_R(M_\sigma)$, must be a division ring, and furthermore, it must be quasi-injective in $R$–Mod since it is quasi-injective in $R$–Mod/$\sigma$.

(c)⇒(d). If $M_\sigma$ is quasi-injective, then $M \subseteq M \subseteq M_\sigma$, so each endomorphism of $M$ can be extended uniquely to an endomorphism of $M_\sigma$. Since $M$ is invariant in $M_\sigma$, $\text{End}_R(M) = \text{End}_R(M_\sigma)$.

(d)⇒(a). Every nonzero homomorphism from a submodule of $M$ into $M$ extends to an endomorphism of $M$; and if $\text{End}_R(M)$ is a division ring, then this extension must be a monomorphism.

The various conditions of the above proposition are found in [6], [15] and [19]. These papers also study a primary decomposition for Noetherian modules over an arbitrary ring, in which the role of prime ideals is played by the prime torsion radicals. A module $RM$ is called rationally complete if $M = M_\sigma$ for $\sigma = \text{rad}_{E(M)}$. In the above
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proposition $M_\sigma$ is rationally complete and monoform, and $E(M_\sigma)$ also defines $\sigma$. Since the quotient category $R{-}\text{Mod}/\sigma$ has an injective cogenerator which is the injective envelope of a simple object, it has a unique simple object, up to isomorphism. This shows that there is a one-to-one correspondence between prime torsion radicals and isomorphism classes of rationally complete monoform modules.

§3. Perfect quotient functors

**THEOREM (3.1).** The following conditions are equivalent for a torsion radical $\sigma$:

(a). The inclusion $V_\sigma : R{-}\text{Mod}/\sigma \rightarrow R_\sigma{-}\text{Mod}$ is an equivalence.

(b). $V_\sigma Q_\sigma$ is naturally isomorphic to $R_\sigma \otimes_R -$.

(c). The inclusion $U_\sigma : R{-}\text{Mod}/\sigma \rightarrow R{-}\text{Mod}$ is exact and preserves direct sums.

(d). Every $R_\sigma$-module is $\sigma$-torsionfree.

(e). $R_\sigma D = R_\sigma$ for each $D \in D_\sigma$.

(f). $R_\sigma$ is a flat right $R$-module, the canonical ring homomorphism $R \rightarrow R_\sigma$ is an epimorphism in the category of rings, and the canonical homomorphism $R_\sigma \otimes_R D \rightarrow R$ is an isomorphism for each $D \in D_\sigma$.

Proof. A circular argument will be used to show that (a) - (d) are equivalent, and then it will be shown that (a), (b) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (d).

The diagram at the right illustrates the situation.

(a) $\Rightarrow$ (b). If $V_\sigma$ is an equivalence, then $V_\sigma Q_\sigma$ is a left adjoint for the inclusion $R_\sigma{-}\text{Mod} \rightarrow R{-}\text{Mod}$ since $Q_\sigma$ is a left adjoint for $U_\sigma$, and so $V_\sigma Q_\sigma$ is naturally isomorphic to $R_\sigma \otimes_R -$ by the uniqueness of adjoint functors.

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(b) ⇒ (c). If $V\sigma Q\sigma \simeq R\sigma \otimes_R -$ , then $V\sigma Q\sigma$ must be right exact and preserve direct sums, and from this it can be shown that $V\sigma$, and hence $U\sigma$, is exact and preserves direct sums.

(c) ⇒ (d). For $M \in R{-}\text{Mod}/\sigma$, let $F_1 \xrightarrow{g} F_0 \xrightarrow{f} M \to 0$ be a free resolution of $M$ in $R\sigma{-}\text{Mod}$. If $U\sigma$ preserves direct sums, then $F_1, F_0 \in R{-}\text{Mod}/\sigma$, and if $U\sigma$ is right exact, then $g : F_1 \to E\sigma(\text{Im}(g))$ must be epic in $R\sigma{-}\text{Mod}$ since it is epic in $R{-}\text{Mod}/\sigma$. Thus $\ker(f) = \text{Im}(g) \in R{-}\text{Mod}/\sigma$, and $M$ is $\sigma$-torsionfree.

(d) ⇒ (a). For $M \in R\sigma{-}\text{Mod}$, $E\sigma(M) \in R{-}\text{Mod}/\sigma$ if $M$ is $\sigma$-torsionfree, and so $E\sigma(M)/M$ is an $R\sigma$-module which is by assumption $\sigma$-torsionfree. This implies that $M = E\sigma(M) \in R{-}\text{Mod}/\sigma$.

(a), (b) ⇒ (f). If (a) and (b) hold, then $R\sigma \otimes_R -$ is exact, and so $(R\sigma)_R$ is flat. The inclusion $R\sigma{-}\text{Mod} \to R{-}\text{Mod}$ must be a full functor since $V\sigma$ is an equivalence and $U\sigma$ is full, and this shows that $R \to R\sigma$ is an epimorphism in the category of rings. (In fact, by [9] the conditions are equivalent.) If $D \in \mathcal{D}_\sigma$, then $Q\sigma(R/D) = 0$, and so $V\sigma Q\sigma(D) \to V\sigma Q\sigma(R)$ is an isomorphism, which implies by (b) that $R\sigma \otimes_R D \to R\sigma \otimes_R R \to R\sigma$ is an isomorphism.

(f) ⇒ (c). $R\sigma D$ is the image in $R\sigma$ of $R\sigma \otimes_R D \to R$.

(c) ⇒ (d). If $M \in R\sigma{-}\text{Mod}$, and $Dm = 0$ for some $m \in M$ and $D \in \mathcal{D}_\sigma$, then $R\sigma m = R\sigma Dm = 0$ implies $m = 0$, and thus $M$ is $\sigma$-torsionfree.

The torsion radical $\sigma$ or the quotient functor $Q\sigma$ is said to be perfect ([18]) if the conditions of the theorem are satisfied. It seems logical to say that $Q\sigma$ is hereditary if it preserves projective objects, and Noetherian if it preserves finitely generated objects.

Certain parts of Theorem 3.1 are proved by Gabriel [5] and Maranda [12], and essentially all of the conditions are studied by the Walkers.
The next few propositions are for the most part well-known. Popescu and Spiru [16] give a converse to condition (f), in that any right flat epimorphic image of $R$ is a ring of quotients. In particular, if $R \rightarrow R'$ is an epimorphism in the category of rings and $R'_R$ is flat, then $R'\text{Mod}$ is a full abelian subcategory of $R\text{Mod}$ whose inclusion functor has an exact left adjoint $R' \otimes_R -$, so it is a quotient category of $R\text{Mod}$, and $R'$ is isomorphic to the ring of quotients determined by a perfect torsion radical. This leads to the following proposition [18].

**Proposition (3.2).** There is a one-to-one correspondence between equivalence classes of ring epimorphisms $R \rightarrow R'$ such that $R'_R$ is flat, and perfect torsion radicals of $R\text{Mod}$.

The functor $U_\sigma$ is naturally isomorphic to $\text{Hom}_R(R_\sigma, -)$, and so $U_\sigma$ is exact iff $R_\sigma$ is projective in $R\text{Mod}/\sigma$ and $U_\sigma$ preserves direct sums iff $R_\sigma$ is small in $R\text{Mod}/\sigma$. Most parts of the proofs of the following three propositions follow immediately from this remark and Propositions 2.3, 2.5, 2.6 and 2.7.

**Proposition (3.3).** The following conditions are equivalent:

(a). $Q_\sigma$ is hereditary.

(b). $U_\sigma$ is exact.

(c). $R_\sigma$ is projective in $R\text{Mod}/\sigma$.

(d). Each $\sigma$-dense left ideal of $R$ contains a $\sigma$-projective, $\sigma$-dense left ideal.

Proof. Since $Q_\sigma$ is a left adjoint for $U_\sigma$, it follow that (b) ⇒ (a). It is clear that (a) ⇒ (c) ⇒ (b) and (c) ⇔ (d).
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PROPOSITION (3.4). The following conditions are equivalent:

(a). $Q_\sigma$ is Noetherian.
(b). $R_\sigma$ is finitely generated in $R\text{-Mod}/\sigma$.
(c). Each $\sigma$-dense left ideal of $R$ contains a finitely generated $\sigma$-dense left ideal.

Proof. Clearly (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). Assume that (c) holds, and that $M \in R\text{-Mod}$ is finitely generated, say $M = \sum_{i=1}^n Rm_i$, with $m_i \in M$. If $M' \subseteq M$ is $\sigma$-dense, then $A_i = \{ r \in R \mid rm_i \in M \}$ is $\sigma$-dense, so it contains a finitely generated left ideal $D_i \in \mathcal{D}_\sigma$, and so $\sum_{i=1}^n D_im_i$ is a finitely generated $\sigma$-dense submodule of $M'$. This shows that $Q_\sigma(M)$ is finitely generated.

PROPOSITION (3.5). The following conditions are equivalent:

(a). $Q_\sigma$ preserves small objects.
(b). $U_\sigma$ preserves direct sums.
(c). $R_\sigma$ is small in $R\text{-Mod}/\sigma$.
(d). If $\{A_i\}_{i=1}^\infty$ is an ascending chain of left ideals such that $\bigcup_{i=1}^\infty A_i \in \mathcal{D}_\sigma$, then $A_i \in \mathcal{D}_\sigma$ for some $i$.

Proof. Clearly (a) $\Rightarrow$ (c) $\Rightarrow$ (b), (c) $\Leftrightarrow$ (d), and (b) $\Rightarrow$ (a) follows easily from the fact that $Q_\sigma$ is a left adjoint for $U_\sigma$.

If $R$ is left hereditary and left Noetherian, then the quotient functor $Q_\sigma$ must be hereditary and Noetherian. In fact, using the conditions on $R_\sigma$ as an object in $R\text{-Mod}/\sigma$ viewed as a quotient category of $R_\sigma\text{-Mod}$ shows that $Q_\sigma$ is hereditary if the ring $R_\sigma$ is left hereditary, and that $Q_\sigma$ is Noetherian if the ring $R_\sigma$ is left Noetherian. If $Q_\sigma$ is Noetherian, then it must preserve small objects, and so the following proposition holds.
PROPOSITION (3.6). The following conditions are equivalent:

(a). \( Q_\sigma \) is perfect.

(b). \( Q_\sigma \) is hereditary and Noetherian.

(c). \( Q_\sigma \) is a small projective object in \( R-\text{Mod}/\sigma \).

PROPOSITION (3.7). Let \( W \in R-\text{Mod} \) be injective, with \( \sigma = \text{rad}_W \).

(a). \( Q_\sigma \) is perfect \( \iff \) \( W \) is a cogenerator in \( R_\sigma-\text{Mod} \).

(b). If \( \sigma \) is complete, then \( Q_\sigma \) is perfect \( \iff \) \( R_\sigma \) contains a copy of each simple \( R_\sigma \)-module.

Proof. (a). If \( Q_\sigma \) is perfect, then \( R_\sigma-\text{Mod} \) is equivalent to \( R-\text{Mod}/\sigma \), and \( W \) is a cogenerator in \( R-\text{Mod}/\sigma \) since \( \sigma = \text{rad}_W \). If \( W \) is a cogenerator in \( R_\sigma-\text{Mod} \), then every \( R_\sigma \)-module is \( \sigma \)-torsionfree and hence \( Q_\sigma \) is perfect.

(b). If \( \sigma \) is complete, then \( \sigma = \text{rad}_W \) for \( W = E(R/\text{rad}_\sigma(R)) = E(R_\sigma) \), and \( E(R_\sigma) \) is a cogenerator in \( R_\sigma-\text{Mod} \) iff it contains a copy of each simple \( R_\sigma \)-module.

The next proposition characterizes perfect prime torsion radicals. A commutative ring has a unique maximal ideal iff it has a unique simple module (up to isomorphism), and this motivates the definition of a prime torsion radical as one for which the quotient category has a unique simple object (up to isomorphism). To obtain any useful conditions on the ring of quotients it appears to be necessary to assume that the torsion radical is perfect.

PROPOSITION (3.8). Let \( M \in R-\text{Mod} \), with \( M \) monoform and rationally complete, and let \( \sigma = \text{rad}_{E(M)} \).

(a). \( \sigma \) is a perfect prime \( \iff \) \( R_\sigma/J(R_\sigma) \) is a simple ring and \( M \) is the only simple \( R_\sigma \)-module.
(b). If \( \sigma \) is a perfect prime, then \( R_\sigma/J(R_\sigma) \) is simple Artinian \( \iff \) \( M \) is finitely generated over its endomorphism ring.

Proof. Since \( M \) is rationally complete, \( M = M_\sigma \). The Jacobson radical of \( R_\sigma \) is denoted by \( J(R_\sigma) \).

(a). If \( \sigma \) is perfect, then \( R-\text{Mod}/\sigma \) is equivalent to \( R_\sigma-\text{Mod} \). This shows that if \( \sigma \) is a prime, then \( M \) must be the unique simple \( R_\sigma \)-module (up to isomorphism), and so \( J(R_\sigma) = \text{Ann}(M) \) is a maximal ideal of \( R_\sigma \).

Conversely, if \( R_\sigma/J(R_\sigma) \) is simple, \( R_\sigma \) has only one isomorphism class of simple modules, and \( M \) is simple, then \( E(M) \) is a cogenerator in \( R_\sigma-\text{Mod} \). Therefore \( \sigma \) is perfect, and it must be a prime since \( M \) is simple in \( R-\text{Mod}/\sigma \) and \( \text{End}_R(M) \) is a division ring.

(b). If \( M \) is finitely generated over \( \text{End}_R(M) \), then \( R_\sigma/J(R_\sigma) \) can be embedded in a finite direct sum of copies of \( M \) since \( J(R_\sigma) = \text{Ann}(M) \). This shows that \( R_\sigma/J(R_\sigma) \) is simple Artinian. The converse is clear.

**Proposition (3.9).** If \( \sigma \) is a complete torsion radical, with \( \text{rad}_\sigma(R) = K \), then the following conditions are equivalent:

(a). \( R_\sigma \) is semisimple Artinian.

(b). \( Q_\sigma \) is perfect and \( R/K \) is a nonsingular left \( R/K \)-module.

(a). The module \( R/K \) is nonsingular and has finite uniform dimension.

Proof. (a) \( \iff \) (b). If \( R_\sigma \) is semisimple Artinian, then \( R_\sigma \) is a cogenerator in \( R_\sigma-\text{Mod} \). Since \( \sigma \) is complete, this implies that \( Q_\sigma \) is perfect, and then every object in \( R-\text{Mod}/\sigma \) is injective since \( R-\text{Mod}/\sigma \) is equivalent to \( R_\sigma-\text{Mod} \). By Proposition 2.2, \( R/K \) is nonsingular.
Conversely, if $Q_\sigma$ is perfect and $R_K R/K$ is nonsingular, then $R_\sigma$-$\text{Mod}$ and $R$-$\text{Mod}/σ$ are equivalent and Proposition 2.2 shows that $R_\sigma$ is semisimple Artinian, since every left $R_\sigma$-module is injective.

(b)$⇔$(c). If $R_K R/K$ is nonsingular, then every object in $R$-$\text{Mod}/σ$ is injective, hence projective, and so $Q_\sigma$ is hereditary. Thus $Q_\sigma$ is perfect iff $R_\sigma$ is finitely generated in $R$-$\text{Mod}/σ$, and the proof can be completed by applying Corollary 2.4.

The above result is contained in Gabriel [5], since $σ$ is complete and hence $R_σ = Q_{\text{max}}(R/K)$. Note that as a consequence of Proposition 3.8, in this situation $R_\sigma$ is simple iff $σ$ is a perfect prime.

The final theorem presents a proof (from [2]) of a variant of Goldie’s theorem characterizing orders in semisimple Artinian rings. The proof uses Propositions 2.8 and 3.9, and does not appeal to the Artin-Wedderburn theorem describing semisimple Artinian rings (which, in fact, it contains as a special case).

Several preliminary notes are necessary before proving the theorem. If $R_U$ is uniform and $R_X$ is nonsingular, then it is easy to show that any nonzero homomorphism from $U$ to $X$ must be a monomorphism. In particular, a nonsingular uniform module must be monomorphic. The ring $R$ has finite uniform dimension, with $\dim(R) = n$, iff there is an essential direct sum $U_1 + \cdots + U_n$ of uniform left ideals. In this case, if $0 → R → U^n$ is an exact sequence, then it can be shown that for some $k ≤ n$ there is an exact sequence $0 → R → U^k$.

**THEOREM (3.10).** If $R$ is semiprime and $R_R$ is nonsingular with finite uniform dimension, the $R$ is a left order in a finite direct product of full matrix rings over division rings.

Proof. Assume that $R$ satisfies the hypothesis of the theorem. Then since $R$ has finite uniform dimension it contains an essential finite
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direct sum of uniform left ideals, say $U_1 \oplus \cdots \oplus U_n$, which is faithful since $R$ is semiprime. For any ideal $A$ and left ideal $U_i' \subseteq U_i$, $AU_i' = 0$ implies $AU_i = 0$ since $R$ is semiprime and $U_i$ is uniform. This can be used to show that $\text{Ann}(U_i)$ is a prime ideal, so that $\cap_{j=1}^m P_j = 0$, where $\{P_j\}_{j=1}^m$ is the set of distinct elements of $\{\text{Ann}(U_i)\}_{i=1}^n$. If $P_j = \text{Ann}(U_i)$ and $U_k U_i \neq 0$, then for some $x \in U_i$ the homomorphism $[u \mapsto ux] : U_k \rightarrow U_i$ is nonzero, so since $U_k$ is uniform and $U_i$ is nonsingular, $U_k$ is isomorphic to a submodule of $U_i$, and thus $P_j U_k = 0$. Therefore $P_j U_k \neq 0$ implies $U_k U_i = 0$ and $U_k \subseteq P_j$, so if $A_j$ is the ideal generated by the left ideals of $\{U_i\}_{i=1}^n$ which are annihilated by $P_j$, then $A_j \cap P_j = 0$ and $A_j \oplus P_j$ is essential in $R$. Since $A_j$ projects to an ideal in the prime ring $R/P_j$, it must be essential, so $\oplus_{j=1}^m A_j \subseteq R \subseteq \prod_{j=1}^m R/P_j$, which shows that $R$ must be an essential $R$-submodule of $\prod_{j=1}^m R/P_j$. The second half of the proof will show that $R/P_j$ is a left order in $Q_{\text{max}}(R/P_j)$, which is a full ring of matrices over a division ring, and then it is not difficult to show that $R$ is a left order in $\prod_{j=1}^m Q_{\text{max}}(R/P_j)$. By Proposition 3.9, $R$ satisfies the descending chain condition on left annihilators since it is a subring of a left Artinian ring. Thus $P_j = \text{Ann}(U_i)$ must be minimal among annihilators of finite subsets of $U_i$, and so there exists an exact sequence $0 \rightarrow R/P_j \rightarrow U_i^k$ for some positive integer $k$. This shows that $R/P_j$ is nonsingular, and it can be assumed that $\dim(R/P_j) = k$.

This reduces the proof to the case in which $R$ is prime, nonsingular, and has $\dim(R) = k$, with a uniform left ideal $U$ for which there exists an exact sequence $0 \rightarrow R \rightarrow U^k$. If $\sigma = \text{rad}_{E(U)}$ and $V = U_\sigma$, then $V$ is injective by Proposition 2.2, and so $\text{rad}_{E(R)} = \text{rad}V$. Furthermore, $V$ must then have dimension $k$ over its endomorphism ring $D$, which is a division ring since $U$ is monoform, and since $V$ is finitely generated over $D$, $Q_{\text{max}}(R) = \text{Bic}(R V)$ is the ring of $k \times k$ matrices over $D^{\text{opp.}}$. 425
If \( q \in Q_{\text{max}}(R) \), then \( B = Rq^{-1} \) is an essential left ideal of \( R \), so let \( B_i \) be the intersection of \( B \) and the \( i^{th} \) component of \( U^k \). Using arguments from the first half of the proof, there is an exact sequence

\[ 0 \to R \to B_k^1 \ (\dim(R) = k), \]

and \( B_i \) is isomorphic to a submodule of \( B_i \), for all \( i \). Thus \( R \subseteq B_k^1 \subseteq \bigoplus_{i=1}^k B_i \subseteq B \), and so \( B \) contains a left regular element \( b \) of \( R \). But \( b \) is left regular in the Artinian ring \( Q_{\text{max}}(R) \) since \( R \) is essential, and hence \( b \) is invertible there. If \( bq = a \), then \( q = b^{-1}a \), with \( a, b \in R \), and so \( R \) is a left order in \( Q_{\text{max}}(R) \).

REFERENCES

2. J. A. Beachy and W. D. Blair, Rings whose faithful left ideals are cofaithful, preprint (1974)
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