Examples in Noncommutative Localization

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Examples are given to show that Goldie's localization at a prime ideal need not be Noetherian even for a ring finitely generated as a module over its Noetherian center. © 1986 Academic Press, Inc.

In [3] and [4] Goldie defined a localization $Q$ of a left- and right-Noetherian ring $R$ at a prime ideal $P$ in the following manner. If $P^{(n)}$ is the $n$th symbolic power of $P$, then $R/P^{(n)}$ has an Artinian classical quotient ring $Q_0(R/P^{(n)})$, for each positive integer $n$, and there is a canonical epimorphism $Q_0(R/P^{(n+1)}) \to Q_0(R/P^{(n)})$. Let $Q^*$ be the inverse limit of the rings $\{Q_0(R/P^{(n)})\}_{n=1}^{\infty}$ under these epimorphisms, and let $\mu: R \to Q^*$ be the induced homomorphism. The Goldie localization $Q$ is defined as the intersection of all subrings $T$ of $Q^*$ such that

(i) $\mu(R) \subseteq T$ and $\mu(P) \subseteq J(T) \subseteq J(Q^*)$ for the Jacobson radical $J(T)$

and

(ii) $T/J(T)$ is simple Artinian and $\bigcap_{n=1}^{\infty} J(T)^n = (0)$. In [3] Goldie asked whether the localization $Q$ he defined at a prime ideal $P$ of a Noetherian ring $R$ must also be Noetherian. Our first example is of a ring $R$ which is finitely generated as a module over its Noetherian center for which the localization $Q$ at a maximal ideal fails to be either right or left Noetherian. Our second example is one where the localization $Q$ is Noetherian on one side only.

Small and Stafford [5] have constructed several interesting noncommutative rings utilizing an example of Nagata. Our second example is one of the rings constructed by Small and Stafford [5] and our first example is inspired by their idea.

Nagata has constructed a pair of commutative Noetherian domains $A \subset B$ such that $A$ is a local ring with maximal ideal $I$, and $B$ is finitely generated as a module over $A$, with two maximal ideals $P, P'$ such that $P \cap P' = I$, and $B/P \cong A/I$. See Zariski–Samuel [6] for a discussion of this example. We fix this notation.
As a final preliminary we recall a few facts about the inversive localization $R_{\Gamma(P)}$ of $R$ at the prime ideal $P$, introduced by Cohn in [2]. If $\Gamma(P)$ denotes the set of matrices (of all sizes) regular modulo $P$, then $R_{\Gamma(P)}$ is the ring universal with respect to the property that all matrices in $\Gamma(P)$ are invertible over $R_{\Gamma(P)}$. Beachy [1] has shown that if $P$ is a prime ideal of the Noetherian ring $R$ with inversive localization $R_{\Gamma(P)}$ and Goldie localization $Q$ at $P$ then $Q \approx R_{\Gamma(P)}$ if any only if $\bigcap_{n=1}^{\infty} J(R_{\Gamma(P)})^n = (0)$.

**Example 1.** Let $R$ denote the set of all matrices of the form

$$
\begin{bmatrix}
  b & 0 & 0 \\
  x & a & 0 \\
  y & z & c
\end{bmatrix},
$$

where $b, c \in B$, $a \in A$, $x, y, z \in P$ and $b-a, c-a \in P$. Here $A, B, P$ are the domains and maximal ideal, respectively, of Nagata's example. Let $M$ be the maximal ideal of $R$ consisting of all matrices of the form

$$
\begin{bmatrix}
  b & 0 & 0 \\
  x & a & 0 \\
  y & z & c
\end{bmatrix},
$$

where $b, x, y, z, c \in P$ and $a \in I$. The ideal $M$ is the kernel of the natural projection from $R$ onto $A/I$ and so $R/M$ is a field. It is easy to see that $R$ is right and left Noetherian since it is finitely generated as a module over its center $A$.

Let $\mathcal{C}(M)$ denote the set of elements of $R$ which are regular modulo $M$. It is clear that

$$
\begin{bmatrix}
  b & 0 & 0 \\
  x & a & 0 \\
  y & z & c
\end{bmatrix} \in \mathcal{C}(M)
$$

if and only if $a \notin I$. Also note that

$$
\begin{bmatrix}
  b & 0 & 0 \\
  x & a & 0 \\
  y & z & c
\end{bmatrix} = \begin{bmatrix}
  ba^{-1} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  a & 0 & 0 \\
  x & a & 0 \\
  y & z & a
\end{bmatrix} \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & ca^{-1}
\end{bmatrix}
$$

and that

$$
\begin{bmatrix}
  a & 0 & 0 \\
  x & a & 0 \\
  y & z & a
\end{bmatrix}^{-1} = \begin{bmatrix}
  a^{-1} & 0 & 0 \\
  -xa^{-2} & a^{-1} & 0 \\
  xza^{-3} - ya^{-2} & -za^{-2} & a^{-1}
\end{bmatrix}
$$
when $a \neq I$. Thus to invert elements of $\mathcal{C}(M)$ we may restrict ourselves to matrices of the form

$$\begin{bmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix},$$

where $d - 1 \in P$.

Set $\Sigma = \{d \in B \mid d - 1 \in P\}$. Let $R_{\Sigma}$ be the ring of all matrices

$$\begin{bmatrix} bd^{-1} & 0 & 0 \\ xd^{-1} & a & 0 \\ yd^{-1} & zd^{-1} & cd^{-1} \end{bmatrix},$$

where $b, c \in B, a \in A, x, y, z \in P, d \in \Sigma$ and $bd^{-1} - a, cd^{-1} - a \in P_{\Sigma}$. Since

$$\begin{bmatrix} P_{\Sigma} & 0 & 0 \\ P_{\Sigma} & I & 0 \\ P_{\Sigma} & P_{\Sigma} & P_{\Sigma} \end{bmatrix}$$

is a quasiregular maximal ideal of $R_{\Sigma}$, it is the Jacobson radical, $J(R_{\Sigma})$, of $R_{\Sigma}$. It is easy to verify that $\bigcap_{n=1}^{\infty} J(R_{\Sigma})^{n} = (0)$, since $\bigcap_{n=1}^{\infty} P_{\Sigma}^{n} = (0)$.

Let $D$ be an $n \times n$ matrix over $R$ which is regular modulo $M$. Since $M_{n}(R)/M_{n}(M) \cong M_{n}(R/M) \cong M_{n}(R_{\Sigma}/M_{\Sigma}) \cong M_{n}(R_{\Sigma})/J(M_{n}(R_{\Sigma}))$, $D$ is invertible in $M_{n}(R_{\Sigma})$. Thus, to show that $R_{\Sigma}$ is the inversive localization of $R$ with respect to $M$, we must show that given any ring $T$ and any homomorphism $\phi: R \to T$ such that $\phi(D)$ is invertible over $T$ for any matrix $D$ which is regular modulo $M$, there exists a unique extension $\bar{\phi}: R_{\Sigma} \to T$ such that $\bar{\phi} |_{R} = \phi$. Let such a homomorphism $\phi$ be given and suppose that

$$X = \begin{bmatrix} bd^{-1} & 0 & 0 \\ xd^{-1} & a & 0 \\ yd^{-1} & zd^{-1} & cd^{-1} \end{bmatrix} \in R_{\Sigma}.$$

Let $X = AD^{-1} + B$ for the matrices

$$A = \begin{bmatrix} b & 0 & 0 \\ x & a & 0 \\ y & 0 & c \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix},$$

where $D$ is regular modulo $M$. Define $\bar{\phi}$ by $\bar{\phi}(X) = \phi(A) \phi(D)^{-1} + \phi(D)^{-1} \phi(B)$. It is not hard to show that $\bar{\phi}$ is well defined.
and additive. To show that \( \phi \) is multiplicative we note the following "partial" Ore conditions in \( R \). We have \( DB = (DB)D \) and \( AD = DA' \), for

\[
A' = \begin{bmatrix}
    b & 0 & 0 \\
    dx & a & 0 \\
    y & 0 & c
\end{bmatrix}.
\]

Using the above notation, with subscripts, for matrices \( X_1 = A_1 D_1^{-1} + D_1^{-1} B_1 \) and \( X_2 = A_2 D_2^{-1} + D_2^{-1} B_2 \), we have

\[
\phi(X_1) \phi(X_2) = \phi(A_1 A_2') \phi(D_1 D_2)^{-1} + \phi(A_1) \phi(D_1 D_2)^{-1} \phi(B_2) + \phi(D_1)^{-1} \phi(B_1 A_2) \phi(D_2)^{-1} + \phi(D_1)^{-1} \phi(B_1) \phi(B_2).
\]

Expanding the last three terms we have

\[
\phi(A_1) \phi(D_1 D_2)^{-1} \phi(B_2)
= \phi \left( \begin{bmatrix} b_1 & 0 & 0 \\ x_1 & a_1 & 0 \\ y_1 & 0 & c_1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix} \right)^{-1}
\times \left( \begin{bmatrix} d_1 d_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{bmatrix} \right)
= \phi \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix} \right) \phi \left( \begin{bmatrix} b_1 & 0 & 0 \\ x_1 & a_1 & 0 \\ y_1 d_1 d_2 & 0 & c_1 \end{bmatrix} \right) \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{bmatrix} \right)
\times \left( \begin{bmatrix} d_1 d_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1}
= \phi(D_1 D_2)^{-1} \phi \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_1 z_2 & 0 \end{bmatrix} \right),
\]

\[
\phi(D_1)^{-1} \phi(B_1 A_2) \phi(D_2)^{-1}
= \phi \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_1 x_2 & 0 & 0 \end{bmatrix} \phi(D_1 D_2)^{-1} + \phi(D_1 D_2)^{-1} \phi \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & d_2 z_1 a_2 & 0 \end{bmatrix} \right)
and \( \phi(B_1, B_2) = 0 \). Collecting terms, we obtain \( \phi(X_1) \phi(X_2) = \phi(X_1, X_2) \). It is routine to check that \( \phi \) is an extension of \( \phi \). By the result of Beachy [1], \( R_X \) is the Goldie localization of \( R \) at the maximal ideal \( M \).

To see that \( R_X \) is not left Noetherian, consider the left \( R_X \)-module

\[
\begin{bmatrix}
0 & 0 & 0 \\
\Sigma & 0 & 0 \\
\Sigma & 0 & 0 \\
\end{bmatrix}
\] / \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\Sigma & 0 & 0 \\
\end{bmatrix}
\]

which has the structure of the left \( A \)-module \( P_{\Sigma} \). If \( P_{\Sigma} \) were left Noetherian as \( A \)-module, this would force \( B_{\Sigma} \) to be Noetherian as a \( B \)-module since \( B_{\Sigma}/P_{\Sigma} \approx B/P \). This would, in turn, force \( B_{\Sigma} = B \), which is impossible, since there exists an element \( c \) in the prime ideal \( P' \) such that \( c - 1 \in P \), and then \( c \) is invertible in \( B_{\Sigma} \) but not in \( B \). By considering the right \( R_X \)-module

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\Sigma & \Sigma & 0 \\
\end{bmatrix}
\] / \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\Sigma & 0 & 0 \\
\end{bmatrix}
\]

we see that \( R \) is not right Noetherian.

**EXAMPLE 2.** We continue with the notation of the Nagata example. Let \( R \) be the set of all matrices of the form \( \begin{bmatrix} b & 0 \\ x & a \end{bmatrix} \), where \( b \in B, x \in P, a \in A \) and \( b - a \in P \). Let \( M \) be the maximal ideal of \( R \) consisting of all matrices of the form \( \begin{bmatrix} b & 0 \\ x & a \end{bmatrix} \) such that \( b, x \in P \) and \( a \in I \). The ideal \( M \) is the kernel of the natural projection from \( R \) onto \( A/I \), and so \( R/M \) is a field.

Let \( \mathcal{C}(M) \) denote the set of elements regular modulo \( M \). Then \( \begin{bmatrix} b & 0 \\ x & a \end{bmatrix} \in \mathcal{C}(M) \) if and only if \( a \notin I \). Note that

\[
\begin{bmatrix} b & 0 \\ x & a \end{bmatrix} = \begin{bmatrix} ba^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ x & a \end{bmatrix},
\]

and that the second matrix is invertible. Thus to check that \( \mathcal{C}(M) \) is a right Ore set it is only necessary to check that the right Ore condition holds for matrices of the form \( \begin{bmatrix} b & 0 \\ x & a \end{bmatrix} \), where \( b - 1 \in P \). To construct the right ring of fractions of \( R \) it is only necessary to invert matrices of this form.

Let \( \Sigma = \{ d \in B \mid d - 1 \in P \} \). The right ring of fractions \( R_M \) of \( R \) at \( M \) is the ring \( S \) of all matrices of the form \( \begin{bmatrix} bd^{-1} & 0 \\ xd^{-1} & a \end{bmatrix} \), where \( bd^{-1} \in B_{\Sigma}, xd^{-1} \in P_{\Sigma}, a \in A \) and \( bd^{-1} - a \in P_{\Sigma} \). Each element of \( \mathcal{C}(M) \) of the form \( \begin{bmatrix} b & 0 \\ x & a \end{bmatrix} \), \( b - 1 \in P \) is invertible in \( S \) since if \( b - 1 \in P \) then \( b^{-1} - 1 = b^{-1}(1 - b) \in P_{\Sigma} \). For \( \begin{bmatrix} bd^{-1} & 0 \\ xd^{-1} & a \end{bmatrix} \in S \), we have

\[
\begin{bmatrix} bd^{-1} & 0 \\ xd^{-1} & a \end{bmatrix} = \begin{bmatrix} b & 0 \\ x & a \end{bmatrix}^{-1} \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}^{-1},
\]
where \([\begin{array}{c} b \\ a \end{array} \in R\) since \(bd^{-1} - a \in P\) implies \(b - a \in P\) and then adding \(ad - a = a(d - 1) \in P\) shows that \(b - a \in P\). Thus \(S = R_M\).

The inversive localization of \(R\) at \(M\) coincides with the right Ore localization at \(\mathfrak{C}(M)\) whenever the latter exists, and thus \(R_M\) is the inversive localization of \(R\) at \(M\) and is right Noetherian since the classical localization of a right Noetherian ring is right Noetherian. As in Example 1, the inversive localization \(R_M\) of \(R\) at \(M\) is the Goldie localization of \(R\) at \(M\) since \(\bigcap_{i=1}^{\infty} H(R_M)^n = (0)\). To see that \(R_M\) is not left Noetherian we consider the left ideal \([\begin{array}{c} 0 \\ P \end{array}]\) of \(R_M\). This has the structure of the left \(A\)-module \(P_X\), and as before, is not Noetherian.

REFERENCES