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SOME ASPECTS OF NONCOMMUTATIVE LOCALIZATION

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This paper is expository in nature, although several results (including 1.5, 1.6(2), 2.7 and 3.1) appear to be new, and continues the account of localization for noncommutative rings begun in [4]. In recent papers, Lambek and Michler [15, 16] and Jategaonkar [13] have investigated the localization at a prime or semiprime ideal of a Noetherian ring. By studying the localization at a semiprime Goldie ideal, as in [7], some of their results can be extended to any ring with Krull dimension (see [9] for the definition of Krull dimension). An ideal I of a right R will be called semiprime (prime) Goldie if the ring R/I is a semiprime (prime) ring with a.c.c. on left annihilators and finite uniform dimension on the left. Any semiprime ideal of a ring with Krull dimension is of this type [9, Corollary 3.4]. The approach in section 3 also makes it possible to extend Small's theorem [19] to give conditions under which the localization at a semiprime Goldie ideal is Artinian.

In Theorem 3.1, the module theoretic characterization of the torsion radical determined by a prime Goldie ideal uses the notion of a strongly prime module, which was introduced in [6]. Section 1 contains a characterization of strongly prime modules, some results on strongly prime rings, and a torsion theoretic characterization of semiprime left Goldie rings. Maximal torsion radicals, used in the latter characterization, are studied in section 2. The Walkers [22] have shown that for a commutative Noetherian ring, maximal torsion radicals correspond to minimal prime ideals, and this result can also be extended [2] to rings with Krull dimension (on either side). In the interests of simplicity, this result, as well as the extension of Small's theorem, is stated for a left Noetherian ring, although the proof is valid in the more general setting.

Throughout the paper, R–Mod will denote the category of unital left R-modules over an associative ring R with identity. A functor σ : R–$Mod \to R$–Mod is called a torsion preradical if for all modules ${}_RM$, ${}_RN$ and all $f \in \mathrm{Hom}_R(M, N)$, $\sigma M \subseteq M$, $f(\sigma M) \subseteq \sigma N$, (i.e. σ is a subfunctor of the identity) and $\sigma M' = \sigma M \cap M'$ for all submodules $M' \subseteq M$. The submodule σM will be denoted by $\mathrm{rad}_\sigma(M)$. The functor σ is called a torsion radical if in addition $\mathrm{rad}_\sigma(M/\mathrm{rad}_\sigma(M)) = 0$ for all $M \in R$–Mod. A module ${}_RM$ is called σ-torsion if $\mathrm{rad}_\sigma(M) = M$ and σ-torsionfree if $\mathrm{rad}_\sigma(M) = 0$; a submodule M' is σ-dense if M/M' is σ-torsion and σ-closed if M/M' is σ-torsionfree. The σ-closure of M' in M is defined as the intersection of all σ-closed submodules of M which contain M'.

We will use the notation $\sigma \leq \tau$ for torsion preradicals σ, τ such that $\mathrm{rad}_\sigma(M) \subseteq \mathrm{rad}_\tau(M)$ for all $M \in R$–Mod. A torsion radical is called maximal if it is proper (not the identity functor) and is maximal with respect to the relation $\leq$. If ${}_RW$ is an injective module, then for any module ${}_RM$ let $\mathrm{rad}_W(M)$ be the intersection of all kernels of homomorphisms from M into W. Then rad_W defines a torsion radical of R–Mod, and every torsion radical of R–Mod is of this form. The torsion radical $\mathrm{rad}_{E(N)}$ is the largest torsion radical σ for which N is σ-torsionfree, where $\mathrm{E}(N)$ denotes the injective envelope of N. Thus for any torsion radical σ, $\mathrm{rad}_\sigma(N) = 0$ iff $\sigma \leq \mathrm{rad}_{E(N)}$.

For the torsion radical $\sigma = \mathrm{rad}_W$, where W is an injective module, a left ideal $A \subseteq R$ is σ-closed iff A is the left annihilator of a subset of W. In particular, the ideal $K = \mathrm{rad}_\sigma(R)$ is the left annihilator of W. An ideal K is called a torsion ideal if it is the left annihilator of an injective module, and in this case $K = \mathrm{Ann}(\mathrm{E}(R/K))$. This shows that the R-injective envelope of R/K coincides with the R/K-injective envelope of R/K, since in general, if M is an R/I-module for some ideal I, then the R/I-injective envelope of M is $\mathrm{E}({}_{R/I}M) = \{x \in \mathrm{E}({}_RM) \mid Ix = 0\}$.

For a torsion radical σ, the full subcategory determined by all modules ${}_RM$ such that $\mathrm{E}(M)$ and $\mathrm{E}(M)/M$ are σ-torsionfree is called the quotient category determined by σ, and will be denoted by R–Mod/σ. The inclusion functor from $R\text{–}Mod/\sigma \to R\text{–}Mod$ has a left adjoint, denoted by Q_σ, and defined by letting $Q_\sigma(M)$ be the σ-closure of $M/\mathrm{rad}_\sigma(M)$ in $\mathrm{E}(M)/\mathrm{rad}_\sigma(M))$. The module of quotients $Q_\sigma(M)$ will be denoted simply by M_σ.

For any module $M \in R\text{–}Mod/\sigma$, and any element $m \in M$, the homomorphism $[r \to rm] : R \to M$ defined by multiplication can be extended uniquely to $\rho_m : R_\sigma \to M$. For any element $q \in R_\sigma$, ρ_q can be used to define right multiplication by q, and this induces a ring structure on R_σ. Furthermore, any module $M \in R\text{–}Mod/\sigma$ becomes a left R_σ-module by defining $qm = \rho_m(q)$, for all $q \in R_\sigma$ and $m \in M$. The ring R_σ is called the ring of quotients determined by σ. The quotient category $R\text{–}Mod/\sigma$ is also a quotient category of $R_\sigma\text{–}Mod$, and the functor Q_σ can be viewed as a functor form $R\text{–}Mod$ to $R_\sigma\text{–}Mod$, although as such it may be only left exact.

The torsion radical σ is called perfect if the following equivalent conditions are satisfied: (1) $R\text{–}Mod/\sigma$ coincides with $R_\sigma\text{–}Mod$, (2) Q_σ is naturally isomorphic to $R_\sigma \otimes_R -$, (3) $R_\sigma D = R_\sigma$ for every σ-dense left ideal $D \subseteq R$. The module ${}_RW$ is an injective cogenerator in $R\text{–}Mod/\sigma$ iff W is injective in $R\text{–}Mod$ and $\sigma = \mathrm{rad}_W$, and it can be shown that σ is perfect iff W is a cogenerator in $R_\sigma\text{–}Mod$.

The torsion radical σ is said to be a prime torsion radical if the quotient category $R\text{–}Mod/\sigma$ has an injective cogenerator which is the injective envelope of a simple object in $R\text{–}Mod/\sigma$, in which case $R\text{–}Mod/\sigma$ has a unique simple object (up to isomorphism). Equivalently, $\sigma = \mathrm{rad}_{E(M)}$ for a monoform module M. (M is monoform if every nonzero homomorphism from a submodule of M into M is a monomorphism, which occurs iff the endomorphism ring of the quasi-injective envelope of M is a division ring [4, Proposition 2.8].) If M is monoform, $\sigma = \mathrm{rad}_{E(M)}$, and $M = M_\sigma$, then σ is perfect iff $R_\sigma/J(R_\sigma)$ is a simple ring and M is the only simple R_σ-module (up to isomorphism). Note that part (a) of [4, Proposition 3.8] should be corrected to include the condition that R_σ has only one isomorphism class of simple modules.

The complete ring of quotients $Q_{\max}(R)$ is defined by the torsion radical $\mathrm{rad}_{E(R)}$. For $\sigma = \mathrm{rad}_{E(R)}$, the terms σ-torsionfree, σ-dense, and σ-closed will be abbreviated to torsionfree, dense, and closed, respectively. A torsion radical σ will be called complete if $\sigma = \mathrm{rad}_{E(R/K)}$ for $K = \mathrm{rad}_\sigma(R)$. In this case, since K is a torsion ideal, $\mathrm{E}(R/K) = \mathrm{E}({}_{R/K}R/K)$ and consequently R_σ can be identified with $Q_{\max}(R/K)$.

The singular submodule $\mathrm{Z}(M)$ of ${}_RM$ is the set of elements of M whose left annihilator is an essential left ideal of R. This defines a torsion preradical, and the Goldie torsion radical G is the smallest torsion radical which contains it. Then $\mathrm{rad}_{E(R)} \leq G$, with equality iff $\mathrm{Z}(R) = 0$, or equivalently, iff $Q_{\max}(R)$ is von Neumann regular.

1 Strongly prime rings and modules

If ρ is a subfunctor of the identity on R–Mod, then setting $\mathrm{rad}_\tau(M) = M \cap \rho(\mathrm{E}(M))$ for all $M \in R$–Mod defines a torsion preradical τ with $\rho \leq \tau$ and $\tau \leq \sigma$ for all torsion preredicals σ such that $\rho \leq \sigma$. For a module ${}_RN$ the smallest torsion preradical τ such that N is τ-torsion will be denoted by Rad^N, and since any sum of homomorphic images of N must be τ-torsion, it can be shown that $\mathrm{Rad}^N(M) = \sum_{\alpha \in J} f_\alpha(N)$, where f_α runs through all homomorphisms in $\mathrm{Hom}_R(N, \mathrm{E}(M))$. With this definition, it is easy to see that Rad^N is the identity functor iff N generates (in the categorical sense) every injective left R-module. That is, $\mathrm{Rad}^N = 1$ iff N is cofaithful. (A module is faithful iff it cogenerates all projectives.) The module N is cofaithful iff there exist $x_1, \ldots, x_m \in N$ such that $\mathrm{Ann}(x_1, \ldots, x_m) = 0$, so that R can be embedded in a finite direct sum N^m of copies of N.

A nonzero module ${}_RM$ is called prime if for any left ideal $A \subseteq R$ and any submodule $N \subseteq M$, $AN = 0$ implies $AM = 0$ or $N = 0$. This is equivalent to the condition that $\mathrm{Ann}(N) = \mathrm{Ann}(M)$ for all nonzero submodules $N \subseteq M$.

Definition 1.1 *A nonzero module ${}_RM$ is called* strongly prime *if M is prime and for each submodule $0 \neq N \subseteq M$ and each element $y \in M$ there* *exist elements* $x_1, \ldots, x_n \in N$ *such that* $\mathrm{Ann}(x_1, \ldots, x_n) \subseteq \mathrm{Ann}(y)$. *The ring* R *is called* left strongly prime *if the module* ${}_RR$ *is strongly prime, and an ideal* $P \subseteq R$ *is called* (left) strongly prime *if* R/P *is left strongly prime.*

Proposition 1.2 *The following conditions are equivalent for a nonzero module* ${}_RM$*:*

(1) M *is strongly prime;*

(2) *For any torsion preradical* τ *of* R*–Mod, either* $\mathrm{rad}_\tau(M) = 0$ *or* $\mathrm{rad}_\tau(M) = M$*;*

(3) M *is contained in every nonzero fully invariant submodule of* $\mathrm{E}(M)$*;*

(4) *For each* $y \in M$ *and* $0 \neq x \in M$ *there exist* $r_1, \ldots, r_n \in R$ *such that* $\mathrm{Ann}(r_1x, \ldots, r_nx) \subseteq \mathrm{Ann}(y)$.

Proof. (1) $\Rightarrow$ (2). If $0 \neq \mathrm{rad}_\tau(M)$ for some torsion preradical τ, then for $y \in M$ there exist $x_1, \ldots, x_n \in \mathrm{rad}_\tau(M)$ such that $\mathrm{Ann}(x_1, \ldots, x_n) \subseteq \mathrm{Ann}(y)$. Thus for $x = (x_1, \ldots, x_n) \in (\mathrm{rad}_\tau(M))^n$, $f : Rx \to Ry$ given by $f(ax) = ay$ for all $a \in R$ is well-defined, so $Ry \subseteq \mathrm{rad}_\tau(M)$ since Rx is τ-torsion. Thus $M = \mathrm{rad}_\tau(M)$.

(2) $\Rightarrow$ (3). If $0 \neq N \subseteq \mathrm{E}(M)$ is fully invariant, then $\mathrm{Rad}^N(\mathrm{E}(M)) = N$, so $\mathrm{Rad}^N(M) = M \cap \mathrm{Rad}^N(\mathrm{E}(M)) = M \cap N \neq 0$. Thus we must have $\mathrm{Rad}^N(M) = M$, and so $M \subseteq N$.

(3) $\Rightarrow$ (4). For $0 \neq x \in M$, let N be the sum in $\mathrm{E}(M)$ of the homomorphic images of Rx. Then N is fully invariant, so by assumption $M \subseteq N$ and thus $y = \sum_{i=1}^n f_i(r_ix)$ for $r_i \in R$ and $f_i \in \mathrm{Hom}_R(Rx, \mathrm{E}(M))$. Therefore $ay = 0$ if $ar_ix = 0$ for all i.

(4) $\Rightarrow$ (1). If $0 \neq N \subseteq M$ and $AN = 0$ for some left ideal $A \subseteq R$, then let $0 \neq x \in N$. By assumption for any $y \in M$ there exist $r_1, \ldots, r_n \in R$ such that $A \subseteq \mathrm{Ann}(r_1x, \ldots, r_nx) \subseteq \mathrm{Ann}(y)$, so $AM = 0$. This shows that M is prime, and the second condition follows immediately. □

Condition 4 of Proposition 1.2 can be used to show that if ${}_RM$

is strongly prime, then any nonzero submodule of M is strongly prime, and any direct sum of copies of M is strongly prime. Condition 3 then shows that M is strongly prime iff its quasi-injective hull $\overline{M}$ is strongly prime, since $\overline{M}$ is the smallest fully invariant submodule of $\mathrm{E}(M)$ which contains M. Furthermore, the proof of (1) $\Rightarrow$ (2) can be used to show that a quasi-injective module is strongly prime iff it is generated (in the categorical sense) by each of its nonzero submodules, and from this it follows that M is strongly prime iff each of its nonzero submodules generates $\overline{M}$.

For the module ${}_RR$, taking $y = 1$ in Definition 1.1 shows that R is left strongly prime iff R is prime and every nonzero left ideal is cofaithful. The later condition implies that R is prime, and is equivalent to the condition that every nonzero ideal of R is cofaithful. The remarks of the preceding paragraph show that R is left strongly prime iff there exists a cofaithful strongly prime left R-module. Furthermore, condition 3 of the next proposition (which follows immediately from Proposition 1.2) shows that R is left strongly prime iff $\mathrm{E}({}_RR)$ is strongly prime.

Proposition 1.3 *The following conditions are equivalent for the ring R:*

(1) *R is left strongly prime;*

(2) $\mathrm{rad}_\tau(R) = 0$ *for every proper torsion preradical τ of R–Mod;*

(3) $\mathrm{E}(R)$ *has no nontrivial fully invariant submodules;*

(4) *If $0 \neq b \in R$, then there exist $r_1, \ldots, r_n \in R$ such that $ar_ib = 0$ for all i implies $a = 0$.*

Strongly prime rings were originally studied by Rubin [17] as absolutely torsion-free rings, using condition 2 of Proposition 1.3. Viola-Prioli [20] showed the equivalence of conditions 1 and 2; Handelman and Lawrence [10] used the name strongly prime for rings which satisfy condition 4, and showed the equivalence of conditions 2 and 4. Proposition 1.2 was used in [6] to show that the endomorphism ring of a finitely generated projective module over a left strongly prime ring is left strongly prime. The following corollary is due to Rubin [17].

Corollary 1.4 *If R is left strongly prime, then* $\mathrm{Z}({}_RR) = 0$ *and* $Q_{\max}(R)$ *is a simple, von Neumann regular ring.*

Proof. The singular submodule defines a torsion preradical of R–Mod so either $\mathrm{Z}(R) = 0$ or $\mathrm{Z}(R) = R$, and the latter is impossible since $\mathrm{Ann}(1)$ is not essential. Since $\mathrm{Z}(R) = 0$, $Q_{\max}(R)$ is von Neumann regular and $Q_{\max}(R) = \mathrm{E}(R)$ as R-modules, which shows that $Q_{\max}(R)$ is simple because $\mathrm{E}(R)$ has no nontrivial fully invariant R-submodules. □

Proposition 1.5 *Let σ be a torsion radical of R–Mod, with* $\mathrm{rad}_\sigma(R) = K$ *and* $\mathrm{rad}_\sigma(\mathrm{E}(R)) = N$. *The following conditions are equivalent:*

(1) *σ is maximal in the set of proper torsion preradicals of R–Mod;*

(2) *σ is complete, and for any ideal $A \supset K$ of R, $N \oplus A/K$ is cofaithful;*

(3) *σ is complete, K is a strongly prime ideal, and there exist $x_1, \ldots, x_m \in N$ such that $K \cap \mathrm{Ann}(x_1, \ldots, x_m) = 0$.*

Proof. (1) $\Rightarrow$ (3). Since $\mathrm{rad}_\sigma(R/K) = 0$, $\sigma \leq \mathrm{rad}_{E(R/K)}$, so by assumption we must have equality, and thus σ is complete. If $A \supset K$, then let τ be the smallest torsion preradical such that $\sigma \leq \tau$ and A/K is τ-torsion. Since $\mathrm{rad}_\sigma(A/K) \neq A/K$, $\sigma \neq \tau$ and so $\tau = 1$ by assumption. It can be shown that $\mathrm{rad}_\tau(\mathrm{E}(R)) = \mathrm{rad}_\sigma(\mathrm{E}(R)) + \mathrm{Rad}^{A/K}(\mathrm{E}(R))$, so there exist elements $x \in N$, $y_1, \ldots, y_m \in A/K$, and $f_1, \ldots, f_m \in \mathrm{Hom}_R(A/K, \mathrm{E}(R))$ such that $x + \sum_{i=1}^m f_i(y_i) = 1 \in R$. If $k \in K$ and $kx = 0$, then $k = 0$ since $ky_i = 0$ for all i, so $K \cap \mathrm{Ann}(x) = 0$. Furthermore, since $\mathrm{rad}_\sigma(\mathrm{E}(R/K)) = 0$, $\mathrm{Rad}^{A/K}\,\mathrm{E}(R/K) = \mathrm{rad}_\tau(\mathrm{E}(R/K)) = \mathrm{E}(R/K)$ which shows that A/K is cofaithful as an R/K-module. Thus every nonzero ideal of R/K is cofaithful and R/K is left strongly prime.

(3) $\Rightarrow$ (2). By assumption if $A \supset K$, then A/K is cofaithful as an R/K-module, so there exist elements $a_1, \ldots, a_n \in A$ such that $ra_i \in K$ for all i implies $r \in K$. If in addition $rx_i = 0$ for $i = 1, \ldots, m$, then

$r \in K \cap \mathrm{Ann}(x_1, \ldots, x_m) = 0$. This shows that $N \oplus A/K$ is cofaithful.

(2) $\Rightarrow$ (1). If τ is a torsion preradical with $\tau \geq \sigma$ and $\mathrm{rad}_\tau(R/K) = 0$, then $\tau \leq \sigma$ and so $\tau = \sigma$. If $\mathrm{rad}_\tau(R/K) \neq 0$, then let $\mathrm{rad}_\tau(R/K) = A/K$, where A is an ideal of R since $\mathrm{rad}_\tau(R/K)$ is fully invariant in R/K. By assumption $N \oplus A/K$ is cofaithful, and so $\tau \geq \mathrm{Rad}^{N \oplus A/K} = 1$. □

Theorem 1.6 ([3], [6], [10], [18]) *The following conditions are equivalent for the ring R:*

(1) *R is semiprime and every faithful left ideal of R is cofaithful;*

(2) *R is semiprime, and for each ideal A of R there exist elements $a_1, \ldots, a_n \in A$ such that $\ell(A) = \ell(a_1, \ldots, a_n)$;*

(3) *R–Mod has finitely many maximal torsion radicals (which correspond to the minimal prime ideals of R), and every proper torsion preradical of R–Mod is contained in one of them;*

(4) *Every proper torsion preradical of R–Mod is contained in a maximal torsion radical;*

(5) *R is a subdirect product of finitely many left strongly prime rings.*

Proof. $\ell(A)$ denotes the left annihilator $\{r \mid rA = 0\}$ of a set A, while $\mathfrak{r}(A)$ denotes the right annihilator.

(1) $\Rightarrow$ (2). If A is an ideal of R, then $\ell(A) \cap A = 0$ since R is semiprime, and $\ell(A) \oplus A$ is faithful. ($B(\ell(A) \oplus A) = 0$ implies $BA = 0$, so $B^2 \subseteq B\ell(A) = 0$ and $B = 0$.) By assumption there exist $x_1, \ldots, x_n \in \ell(A) \oplus A$ such that $\ell(x_1, \ldots, x_n) = 0$, with $x_i = y_i + a_i$, where $y_i \in \ell(A)$ and $a_i \in A$. If $r \in \ell(a_1, \ldots, a_n)$ and $a \in A$, then $arx_i = 0$ for all i since $ry_i \in \ell(A) = \mathfrak{r}(A)$, and so $ar = 0$. Thus $r \in \mathfrak{r}(A) = \ell(A)$, and so $\ell(A) = \ell(a_1, \ldots, a_n)$. Note that $\ell(A) = \mathfrak{r}(A)$ since R is semiprime.

(2) $\Rightarrow$ (3). If $\{A_\alpha\}_{\alpha \in J}$ is any ascending chain of annihilator ideals with $\ell(A_\alpha) \neq 0$ for all α, then by assumption $\ell(\cup_{\alpha \in J} A_\alpha) = \ell(a_1, \ldots, a_n)$

for elements $a_1, \ldots, a_n \in \cup_{\alpha \in J} A_\alpha$, so $\ell(\cup_{\alpha \in J})A = \ell(A_\alpha) \neq 0$ for some $\alpha \in J$. Applying Zorn's Lemma shows that any proper annihilator ideal is contained in a maximal annihilator ideal, which is then a minimal prime ideal.

If τ is any proper torsion radical, then $\text{rad}_\tau(R)$ is not cofaithful, so by assumption $\ell(\text{rad}_\tau(R)) \neq 0$. Thus $\text{rad}_\tau(R) \subseteq P$ for a minimal prime ideal P, by the preceding argument, and $\mathfrak{r}(P) \cap \text{rad}_\tau(R) = 0$. Then $\text{rad}_\tau(\mathfrak{r}(P)) = 0$, and so $P = \ell(\mathfrak{r}(P))$ implies that R/P can be embedded in a direct product of copies of $\mathfrak{r}(P)$. This shows that $\text{rad}_\tau(R/P) = 0$ and thus $\tau \leq \text{rad}_{E(R/P)}$.

Let$P$ be a minimal prime ideal of R. Then $\mu = \text{rad}_{E(R/P}$ is a maximal torsion radical, since if $\mu \leq \sigma$ and $\sigma \neq 1$, then as above, $\sigma \leq \text{rad}_{E(R/P')}$ for a minimal prime ideal P'. Now $\text{rad}_{E(R/P)} \leq \text{rad}_{E(R/P')}$ implies that P' is $\text{rad}_{E(R/P)}$-closed, so P' is the annihilator of a nonzero submodule of $\text{E}(R/P)$. It follows that $P' \subseteq P$, so $P' = P$ and $\mu = \sigma$ since P' is a minimal prime. Thus the minimal prime ideals of R are just the maximal annihilator ideals, and these correspond to the maximal torsion radicals of R–Mod.

If $A \neq B$ are minimal annihilator ideals, then $A \cap B = 0$ since $A \cap B$ is an annihilator ideal. Thus $AB = 0$, and from this it follows that the sum in R of all minimal annihilator ideals is a direct sum, say $\oplus_{\alpha \in J} A_\alpha$. By assumption $\ell(\oplus_{\alpha \in J} A_\alpha) = \ell(a_1, \ldots, a_n)$ for elements $a_1, \ldots, a_n \in \oplus_{\alpha \in J} A_\alpha$, so $\ell(\oplus_{\alpha \in J} A_\alpha) = \ell(\oplus_{i=1}^n A_i)$ for some finite subset $\{A_i\}_{i=1}^n$. If A is a minimal annihilator which is not in the finite subset, then $A(\oplus_{i=1}^n A_i) = 0$, and this implies that $A^2 \subseteq A(\oplus_{\alpha \in J} A_\alpha) = 0$, so $A = 0$.

(3) $\Rightarrow$ (4). Immediate.

(4) $\Rightarrow$ (1). If A is a nonzero ideal of R, the $\text{Rad}^{R/A} \neq 1$, so by assumption $\text{Rad}^{R/A} \leq \mu$ for some maximal torsion radical μ. Since A is $\text{Rad}^{R/A}$-dense, it must be μ-dense, and then A^2 is μ-dense since μ is a torsion radical. But then $\mu \neq 1$ implies that $A^2 \neq 0$, so R is semiprime. If R/A and A are both μ-torsion for a torsion radical μ, then R must be μ-torsion and so $\mu = 1$. Thus $\text{Rad}^{R/A \oplus A} = 1$, since it cannot be

contained in a proper torsion radical, so $R/A \oplus A$ is cofaithful. Hence there exist elements $x_1, \ldots, x_n \in R/A \oplus A$ with $\mathrm{Ann}(x_1, \ldots, x_n) = 0$, where $x_i = y_i + a_i$ for $y_i \in R/A$ and $a_i \in A$. Then $\ell(a_1, \ldots, a_n) \cap A \subseteq \mathrm{Ann}(x_1, \ldots, x_n) = 0$, so $\ell(a_1, \ldots, a_n) \cdot A = 0$. If A is faithful, then $\ell(a_1, \ldots, a_n) = 0$ and A is cofaithful.

(3) $\Rightarrow$ (5). Let $\{\mu_i\}_{i=1}^n$ be the maximal torsion radicals of R–Mod, with corresponding torsion ideals $\{K_i\}_{i=1}^n$. Then by assumption each μ_i is maximal among proper torsion preradicals, so K_i is a strongly prime ideal by Proposition 1.5. If $K = \cap_{i=1}^n K_i$, then $\mathrm{Rad}^{R/K}(R/K_i) = R/K_i$, so $\mathrm{Rad}^{R/K} \not\leq \mu_i$ for all i, and this shows that $\mathrm{Rad}^{R/K} = 1$. It follows that R/K is cofaithful, so $K = 0$, and R is a subdirect product of finitely many left strongly prime rings.

(5) $\Rightarrow$ (1). Assume that $\cap_{i=1}^n P_i = 0$ for strongly prime ideals P_i, and that A is a faithful ldeal of R. Then for each i, there exist elements $a_{a1}, \ldots, a_{ik} \in A$ such that $ra_{ij} \in P_i$ for all j implies $rA \subseteq P_i$. Then $ra_{ij} = 0$ for all i, j implies $rA \subseteq P_i$ for all i, which implies that $rA \subseteq \cap_{i=1}^n P_i = 0$, so $\ell(\{a_{ij}\}) = 0$ and A is cofaithful. Of course, R is semiprime by assumption. □

Condition 1 was studied in [6], and condition 3 was given in [3]. Note that R has d.c.c. on left annihilators iff for each right ideal A of R there exist elements $a_1, \ldots, a_n \in A$ such that $\ell(A) = \ell(a_1, \ldots, a_n)$. Condition 4 is essentially Rubin's condition [18] that every proper torsion preradical determines a proper torsion radical, since an argument using Zorn's Lemma can be given to show that in general any proper torsion preradical is contained in a maximal torsion preradical. Handelman [10] showed that conditions 1 and 5 are equivalent. The following corollary is due to Rubin [18].

Corollary 1.7 *If R satisfies the conditions of Theorem 1.6, then* $\mathrm{Z}({}_R R) = 0$ *and* $Q_{\max}(R)$ *is isomorphic to a finite direct product of simple von Neumann regular rings.*

Proof. If $\{P_i\}_{i=1}^n$ are the minimal prime ideals of R, then their intersection is irredundant. If $x \in R$ and $x \notin P_i$, then $(\cap_{j \neq i} P_j)x \not\subseteq P_i$, and so there exists $r \in \cap_{j \neq i} P_j$ such that $rx \notin P_i$. This shows, as in [10], that as a left R-module, $\oplus_{i=1}^n R/P_i$ is an essential extension of R. It follows that $\mathrm{Z}(R) = 0$, since $\mathrm{Z}(R/P_i) = 0$ for all i by Proposition 1.4, and that $Q_{\max}(R)$ is isomorphic as an R-module to $\oplus_{i=1}^n Q_{\max}(R/P_i)$. The isomorphism is easily seen to be a ring homomorphism, and the desired conclusion follows from Proposition 1.4. □

Using techniques of free algebras, Handelman and Lawrence [11] have shown that every prime ring can be embedded in a left strongly prime ring, and hence in a simple ring. W. D. Blair (in a private communication) has extended this to show that a semiprime ring can be embedded in a ring satisfying the conditions of Theorem 1.6 iff there exists a finite set of primes $p_1, \ldots, p_s \in \mathbb{Z}$ such that if $0 \neq r \in R$ and $nr = 0$ for some $n \in \mathbb{Z}$, then at least one of the primes $p_1, \ldots, p_s$ divides n.

Finally, we show that the addition of finite uniform dimension to the conditions of Theorem 1.6 gives a characterization of semiprime Goldie rings. Note that condition 3, in particular, gives a characterization which is stated entirely in torsion theoretic language.

Theorem 1.8 ([3], [6]) *The following conditions are equivalent for the ring R:*

(1) *R is a semiprime left Goldie ring;*

(2) *R is semiprime, every faithful ideal is cofaithful, and every nonzero ideal contains a uniform left ideal;*

(3) *Every proper torsion preradical of R–Mod is contained in a perfect maximal torsion radical.*

Proof. (1) ⇒ (2). By assumption, R has d.c.c. on left annihilators and finite uniform dimension.

(2) ⇒ (3). If P_i is a minimal prime ideal of R, then the intersection $\cap_{j \neq i} P_j$ over the remaining minimal prime ideals is nonzero,

so it contains a uniform left ideal by assumption. Then R/P_i must contain a uniform left ideal since $\cap_{j\neq i} P_j \cap P_i = 0$, so R/P_i has finite uniform dimension since every nonzero left ideal of R/P_i is cofaithful. By [4, Proposition 3.9], the maximal torsion radical determined by $\mathrm{E}(R/P_i)$ must be perfect, so the desired conclusion follows from Theorem 1.6.

(3) $\Rightarrow$ (1). By assumption R satisfies the conditions of Theorem 1.6, and by [4, Proposition 3.9] each R-module R/P_i, where P_i is a minimal prime ideal, has finite uniform dimension. Thus R must have finite uniform dimension since it is embedded in a finite direct sum of modules each of which has finite uniform dimension. This completes the proof since R is semiprime and $\mathrm{Z}(R) = 0$. □

2 Maximal torsion radicals

Proposition 2.1 *The following conditions are equivalent for a torsion radical σ of R–Mod.*

(1) *σ is a maximal torsion radical;*

(2) *Every nonzero injective object in R–Mod/σ is a cogenerator;*

(3) *σ is complete and every nonzero torsionfree injective left R_σ-module is faithful;*

(4) *σ is complete and every nonzero torsionfree injective left $R/\mathrm{rad}_\sigma(R)$-module is faithful;*

(5) *σ is complete and $\mathrm{rad}_\sigma(R)$ is maximal in the set of proper σ-closed torsion ideals of R.*

Proof. Let $\mathrm{rad}_\sigma(R) = K$ and recall that σ is complete iff $\sigma = \mathrm{rad}_{E(R/K)}$.

(1) $\Rightarrow$ (2). Let W be any nonzero injective object in R–Mod/σ. Then W is injective as an R-module, so $\mathrm{rad}_\sigma(W) = 0$ implies that $\sigma \leq \mathrm{rad}_W$. Since σ is maximal, $\sigma = \mathrm{rad}_W$ and W is a cogenerator in R–Mod/σ by [4, Proposition 2.1].

(2) $\Rightarrow$ (3). Since $\mathrm{E}(R/K)$ is injective in R–Mod/σ, it must be a cogenerator, so $\sigma = \mathrm{rad}_{E(R/K)}$ and σ is complete. If M is torsionfree and injective as an R_σ-module, then it must be isomorphic to a direct

summand of $\prod_{\alpha \in J} \mathrm{E}(R_\sigma)$ for some index set J, and since $\mathrm{E}(R/K) = \mathrm{E}(R_\sigma)$, the latter is injective as an R-module, so M must also be injective as an R-module. Then $\sigma = \mathrm{rad}_M$ since $\mathrm{rad}_\sigma(M) = 0$, and so M is faithful as an R_σ-module.

(3) $\Rightarrow$ (4). Since $\mathrm{E}(R_\sigma) = \mathrm{E}(R/K)$, a module is a torsionfree injective R_σ-module iff it is a torsionfree injective R/K-module. If such a module is faithful as an R_σ-module, then it is faithful as an R/K-module.

(4) $\Rightarrow$ (5). If $K \subseteq \mathrm{rad}_\tau(R) \neq R$ for a torsion radical τ such that $\mathrm{rad}_\tau(R)$ is σ-closed, then $\mathrm{E}(R/\mathrm{rad}_\tau(R))$ is a torsionfree injective left R/K-module. By assumption $\mathrm{E}(R/\mathrm{rad}_\tau(R))$ is a faithful R/K-module, so $\mathrm{rad}_\tau(R) = K$.

(5) $\Rightarrow$ (1). If $\sigma \leq \tau$ for a proper torsion radical τ, then $\mathrm{rad}_\tau(R)$ is σ-closed and $K \subseteq \mathrm{rad}_\tau(R)$, so by assumption $K = \mathrm{rad}_\tau(R)$. This implies that $\sigma = \tau$ because σ is complete. □

Corollary 2.2 *Let σ be a maximal torsion radical of R–Mod. Then σ is a prime torsion radical if either R is left Noetherian or σ is perfect.*

Proof. Since σ is prime iff R–Mod/σ has an injective cogenerator which is the injective envelope of a simple object in R–Mod/σ, it follows from Proposition 2.1 that σ is prime iff R–Mod/σ has a simple object. If R is left Noetherian, then there exists a maximal σ-closed ideal A, and $Q_\sigma(R/A)$ is simple in R–Mod/σ. If σ is perfect, then R–$Mod/\sigma \simeq R_\sigma$–$Mod$ and must contain simple objects. □

Theorem 2.3 ([1]) *Let σ be a perfect torsion radical. Then σ is maximal $\Longleftrightarrow$ $R_\sigma/\mathrm{J}(R_\sigma)$ is simple Artinian and $\mathrm{J}(R_\sigma)$ is right T-nilpotent.*

Proof. If σ is perfect, then R–$Mod/\sigma \simeq R_\sigma$–$Mod$, so σ is maximal iff every nonzero injective R_σ-module is a cogenerator. If S is a simple R_σ-module, then $\mathrm{E}(S)$ is a cogenerator for R_σ–Mod, and so S must be the only simple R_σ-module (up to isomorphism). For any R_σ-module M, $\mathrm{E}(M)$ must be a cogenerator, so $\mathrm{E}(M)$ contains an isomorphic copy of S, and this minimal submodule is contained in M since M is essential in $\mathrm{E}(M)$. Thus every nonzero injective object of R–Mod/σ is a cogenerator iff R_σ–Mod has only one isomorphism class of simple modules, and every module contains a minimal submodule. This occurs iff $R_\sigma/\mathrm{J}(R_\sigma)$ is simple Artinian ($\mathrm{J}(R_\sigma)$ denotes the Jacobson radical of R_σ) and $\mathrm{J}(R_\sigma)$ is right T-nilpotent, since if $R_\sigma/\mathrm{J}(R_\sigma)$ is simple Artinian, then every left R_σ-module contains a minimal submodule iff $\mathrm{J}(R_\sigma)$ is right T-nilpotent. (Recall that an ideal A is right T-nilpotent if for each sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A there exists an integer n such that $a_n a_{n-1} \cdots a_1 = 0$.) □

If the conditions of Theorem 2.3 are satisfied, then R_σ is right perfect, and since there is only one isomorphism class of simple modules, R_σ is isomorphic to the ring of $n \times n$ matrices over a local ring. Theorem 2.3 also gives some insight into Goldie's theorem, since if R is a semiprime, left Goldie ring, then $Q_{\max}(R)$ is isomorphic to the finite direct product of the rings of quotients defined by the maximal torsion radicals of R–Mod. Each maximal torsion radical μ is perfect with $\mathrm{rad}_\mu(R)$ strongly prime, and so R_μ must be simple Artinian since by Proposition 1.4 it is simple and by Theorem 2.3 it is Artinian modulo its Jacobson radical. In general, if R_μ is simple Artinian, then in R–$Mod/\mu \simeq R_\mu$–Mod every nonzero object is a cogenerator, so by Proposition 2.1, μ is maximal. The ring R will be called quasi-local (semi-local) if $R/\mathrm{J}(R)$ is simple Artinian (semisimple Artinian).

Corollary 2.4 *If R is left Noetherian and σ is a perfect torsion radical then σ is maximal $\iff$ R_σ is a quasi-local left Artinian ring.*

Proof. If σ is perfect and R is left Noetherian, then R_σ is left Noetherian. If σ is maximal, then every left R_σ-module contains a minimal submodule, so the ascending socle series of R_σ terminates after a finite number of steps at R_σ. This gives a composition series for R_σ since each term of the series is finitely generated. The converse follows directly from Theorem 2.3. □

Corollary 2.5 *If R is left hereditary and left Noetherian, then a torsion radical σ is maximal $\Longleftrightarrow$ R_σ is simple Artinian.*

Proof. Let $\mathrm{rad}_\sigma(R) = K$. Since R is hereditary and Noetherian, any homomorphic image of a direct sum of copies of $\mathrm{E}({}_{R/K}R/K) = \mathrm{E}({}_R R/K)$ is injective as an R-module and hence as an R/K-module. But any direct sum of injective R/K-modules or homomorphic image of an injective R/K-module is of this form, so R/K is left hereditary and left Noetherian. Thus R/K has finite uniform dimension and zero singular ideal, so if σ is maximal, then it is complete and $R_\sigma = Q_{\max}(R/K)$ must be semisimple Artinian. Hence R_σ is simple Artinian by Theorem 2.3. The converse follows immediately from Theorem 2.3. □

The Walkers [22] showed that the maximal torsion radicals of a commutative Noetherian ring are in one-to-one correspondence with its minimal prime ideals. Theorem 2.8 extends this result to the noncommutative case. The preliminary propositions can be used to show that maximal torsion radicals correspond to minimal prime ideals for any ring with Krull dimension on either the left or right, or for any right perfect ring. The correspondence holds, of course, for any ring which satisfies the conditions of Theorem 1.6, which includes, in the commutative case, any ring with no nilpotent elements and only finitely many minimal prime ideals.

Proposition 2.6 *Every proper torsion radical of R–Mod is contained in a maximal torsion radical, and the maximal torsion radicals of R–Mod correspond to the minimal prime ideals of R $\Longleftrightarrow$ each nonzero injective left R-module contains a submodule whose left annihilator is a minimal prime ideal.*

Proof. $\Rightarrow$). If ${}_RW$ is a nonzero injective module, then rad_W is a proper torsion radical, so by assumption $\mathrm{rad}_W \leq \mathrm{rad}_{E(R/P)}$ for a minimal prime ideal P. Then P must be rad_W-closed, so P is the left annihilator of a submodule of W.

$\Leftarrow$). If σ is a proper torsion radical of R–Mod, then $\sigma = \mathrm{rad}_W$ for a nonzero injective module W. If P is a minimal prime ideal which is the annihilator of a submodule of W, then $\mathrm{rad}_W(R/P) = 0$ shows that $\mathrm{rad}_W \le \mathrm{rad}_{E(R/P)}$. If P is any minimal prime ideal of R and $\mathrm{rad}_{E(R/P)} \le \sigma$ for a proper torsion radical σ, then as above, $\sigma \le \mathrm{rad}_{E(R/P')}$ for some minimal prime ideal P'. But then P' is $\mathrm{rad}_{E(R/P)}$-closed, so it is the annihilator of a nonzero submodule of $\mathrm{E}(R/P)$, which forces $P' \subseteq P$. Since P is minimal we must have $P' = P$, and thus $\sigma = \mathrm{rad}_{E(R/P)}$. It follows that every minimal prime ideal defines a maximal torsion radical of R–Mod, and then every proper torsion radical is contained in a maximal torsion radical. As above, if $\mathrm{rad}_{E(R/P)} = \mathrm{rad}_{E(R/P')}$, where P and P' are prime ideals, then $P = P'$, and this establishes the one-to-one correspondence between minimal prime ideals of R and maximal torsion radicals of R–Mod. □

Proposition 2.7 *Every proper torsion radical of R–Mod is contained in a maximal torsion radical, and the maximal torsion radicals of R–Mod correspond to the minimal prime ideals of R $\Longleftrightarrow$ $R/\mathrm{P}(R)$ satisfies the same condition, for the prime radical* $\mathrm{P}(R)$ *of R, and* $\mathrm{P}(R)$ *is right T-nilpotent.*

Proof. Recall that an ideal I is right T-nilpotent iff $\{m \in M \mid Im = 0\} \ne 0$ for every left R-module M.

$\Rightarrow$). For any nonzero module ${}_RM$, by the previous proposition there is a submodule N of $\mathrm{E}(M)$ whose left annihilator is a minimal prime ideal P of R. Then $M \cap N \ne 0$ and $\mathrm{P}(R) \cdot (M \cap N) = 0$, which shows that $\mathrm{P}(R)$ is right T-nilpotent. If M is an injective $R/\mathrm{P}(R)$-module, then $N \subseteq M$, and so by the previous proposition $R/\mathrm{P}(R)$ satisfies the desired condition.

$\Leftarrow$). If ${}_RW$ is a nonzero injective R-module, then $M = \{x \in W \mid \mathrm{P}(R)x = 0\}$ is nonzero since $\mathrm{P}(R)$ is right T-nilpotent. Since M is an injective $R/\mathrm{P}(R)$-module, by assumption there is a submodule of M whose left annihilator in $R/\mathrm{P}(R)$ is a minimal prime ideal of $R/\mathrm{P}(R)$.

Buy then the left annihilator in R is a minimal prime ideal of R, and the proof can be completed by applying the previous proposition. $\square$

Theorem 2.8 ([2]) *If R is left Noetherian, then every proper torsion radical of R–Mod is contained in a maximal torsion radical, and the maximal torsion radicals of R–Mod are in one-to-one correspondence with the minimal prime ideals of R.*

Proof. If R is left Noetherian, then the prime radical $\mathrm{P}(R)$ is nilpotent and $R/\mathrm{P}(R)$ is a semiprime left Goldie ring. Since $R/\mathrm{P}(R)$ satisfies the conditions of Theorem 1.6, the result follows from Proposition 2.7. $\square$

The condition that every proper torsion radical is contained in a maximal torsion radical and that maximal torsion radicals correspond to minimal prime ideals is Morita invariant. A finite direct product of rings satisfies the condition iff each factor does; if a commutative ring has the property, then so does the ring of polynomials over it. A prime ring has the property iff every nonzero injective module is faithful.

In the ring R of linear transformations of an infinite dimensional vector space, the prime ideal P of all linear transformations of finite rank is a torsion ideal since R is von Neumann regular. The next proposition shows that P defines a maximal torsion radical even though P is not a minimal prime ideal of R. The ring $C[0,1]$ of continuous functions on the interval $[0,1]$ has no maximal annihilator ideals, and so the same condition holds for its complete ring of quotients Q. It was shown in [1] that the torsion radical determined by $\mathrm{E}(Q)$ is not contained in a maximal torsion radical of Q–Mod.

An ideal P which is maximal in the set of proper σ-closed ideals for some torsion radical σ must be a prime ideal. To show this, suppose that $aRb \subseteq P$ with $b \notin P$. If $B = P + RaR$, then $Bb \subseteq P$, so $\mathrm{Hom}_R(R/B, R/P) \neq 0$, which shows that B is not σ-dense. Thus the σ-closure C of B is a proper σ-closed ideal which contains P, a contradiction. Conversely, if P is a prime ideal, $\sigma = \mathrm{rad}_{E(R/P)}$, and C is a proper σ-closed ideal of R, then $C \subseteq P$, so P is a maximal σ-closed ideal. To show this, if $C \not\subseteq P$, then $Cx \neq 0$ for all $0 \neq x \in R/P$ implies that $\mathrm{Hom}_R(R/C, \mathrm{E}(R/P)) = 0$ and C is σ-dense, a contradiction.

Proposition 2.9 *Let σ be a complete torsion radical, with $\mathrm{rad}_\sigma(R) = K$. If K is a prime ideal, the σ is maximal. The converse holds if K is semiprime.*

Proof. Since σ is complete, $\sigma = \mathrm{rad}\,_E(R/K)$. If K is prime, then by the above remarks it is a maximal σ-closed ideal of R, so σ is maximal by Proposition 2.1.

Conversely, assume that K is semiprime. By the above remarks, to show that K is prime it is sufficient to show that K is a maximal σ-closed ideal. It A is a proper σ-closed ideal which contains K, then $\mathrm{Hom}_R(K, \mathrm{E}(R/A)) = 0$ since K is σ-torsion and $\mathrm{E}(R/A)$ is σ-torsionfree. If $0 \neq f \in \mathrm{Hom}_R(A/K, \mathrm{E}(R/A))$, then since $\sigma = \mathrm{rad}_{E(R/K)}$, there exists $g \in \mathrm{Hom}_R(\mathrm{E}(R/A), \mathrm{E}(R/K))$ such that $gf \neq 0$, and consequently there is a left ideal B with $K \subseteq B \subseteq A$ such that $f(B/K) \subseteq R/A$ and $0 \neq gf(B/K) \subseteq R/K$. Let C be the left ideal containing K with $C/K = gf(B/K)$. Then $B \subseteq A$ implies $B \cdot f(B/K) \subseteq B \cdot (R/A) = 0$ and hence $BC \subseteq K$. Since K is a semiprime ideal, this implies that $CB \subseteq K$, and thus $C \cdot gf(B/K) \subseteq gf(C \cdot (B/K)) = 0$, which shows that $C^2 \subseteq K$. Since K is semiprime, this implies that $C \subseteq K$, a contradiction. Thus $\mathrm{Hom}_R(A/K, \mathrm{E}(R/A)) = 0$, and so we must have $\mathrm{Hom}_R(A, \mathrm{E}(R/A)) = 0$, which shows that $A = \mathrm{rad}_{E(R/A)}(R)$, and so A is a σ-closed torsion ideal of R. By Proposition 2.1, K is a maximal σ-closed torsion ideal, and so $A = K$. This completes the proof. □

Proposition 2.10 *Let σ be a complete torsion radical, with $\mathrm{rad}_\sigma(R) = K$. If R_σ is a prime ring, then σ is maximal. Conversely, if σ is maximal, then either $\mathrm{Z}({}_{R/K}R/K) = 0$ or $\mathrm{Z}({}_{R/K}R/K)$ is essential in R/K, and if $\mathrm{Z}({}_{R/K}R/K) = 0$ then R_σ is a prime ring.*

Proof. If R_σ is a prime ring, then by Proposition 2.9, $\mathrm{E}(R_\sigma)$ defines a maximal torsion radical of R_σ–Mod, and so every nonzero torsionfree injective left R_σ-module is faithful by Proposition 2.1, which then shows that σ is maximal.

Conversely, let σ be a maximal torsion radical. Then $\mathrm{rad}_{E(R/K)}$ defines a maximal torsion radical of R/K–Mod, and since $G \geq \mathrm{rad}_{E(R/K)}$ for the Goldie torsion radical of R/K–Mod, we must have either $G = \mathrm{rad}_{E(R/K)}$, in which case $\mathrm{Z}({}_{R/K}R/K) = 0$, or $G = 1$, in which case $\mathrm{Z}({}_{R/K}R/K)$ is essential in R/K. If $\mathrm{Z}({}_{R/K}R/K) = 0$, then R_σ is von Neumann regular since σ is complete. If A is any proper σ-closed ideal of R_σ, then A is a torsion ideal, so $A = 0$ by Proposition 2.1, and then the remarks preceding Proposition 2.9 show that R_σ is a prime ring. □

3 Localization at semiprime Goldie ideals

Lambek and Michler [15, Theorem 3.9] showed that if P is a prime ideal of a left Noetherian ring, then the torsion radical $\sigma = \mathrm{rad}_{E(R/P}$ is defined by a σ-torsionfree indecomposable injective module (which is unique up to isomorphism). Jategaonkar [14] extended this result to prime Goldie ideals. The following theorem gives both necessary and sufficient conditions for a torsion radical to be defined by a prime Goldie ideal, and can thus be viewed as an extension of the module theoretic characterization of prime left Goldie rings given by Faith [8, Theorem 34].

Theorem 3.1 *The torsion radical σ is defined by* $\mathrm{E}(R/P)$ *for a prime Goldie ideal P $\Longleftrightarrow$ σ is a prime torsion radical defined by a monoform, strongly prime, quasi-injective module which is finitely generated over its endomorphism ring.*

Proof. $\Rightarrow$). If P is a prime Goldie ideal of R, then let M be a uniform left ideal of R/P. Since R/P is left strongly prime, M is cofaithful as an R/P-module, and monoform and strongly prime as an R-module. The same conditions hold for the quasi-injective envelope $\overline{M}$ of M, which is an R/P-module, and the elements $m_1, \dots, m_n \in \overline{M}$ such that $\mathrm{Ann}(m_1, \dots, m_n) = P$ serve as generators for $\overline{M}$ over its endomorphism ring. Now $\mathrm{E}(\overline{M}) \subseteq \mathrm{E}(R/P)$, and so $\sigma \leq \mathrm{rad}_{E(\overline{M})}$, while $\mathrm{rad}_{E(\overline{M})} \leq \sigma$ since $\mathrm{Ann}(\overline{M}) = P$.

$\Leftarrow$). Let M be a monoform, strongly prime, quasi-injective module which is finitely generated over its endomorphism ring, and which defines σ. If $P = \mathrm{Ann}(M)$, then $R/P \subseteq M^n$ for some n, since the generators of M over its endomorphism ring can be used to show that M is cofaithful as an R/P-module. Thus R/P is strongly prime and has finite uniform dimension as an R-module, since M is strongly prime and monoform. Since the same conditions hold for R/P as an R/P-module, P is a prime Goldie ideal by Theorem 1.8. The embedding also shows that $\sigma = \mathrm{rad}\, E(M) \leq \mathrm{rad}_{E(R/P)}$, and that $\mathrm{E}({}_{R/P}R/P) \subseteq M^n$, since M is cofaithful and quasi-injective as an R/P-module and therefore injective as an R/P-module. Because M is cofaithful, it generates $\mathrm{E}({}_{R/P}R/P)$, so there exists a nonzero homomorphism $f : M \to \mathrm{E}({}_{R/P}R/P)$, and since M is monoform, f followed by the inclusion into M^n must be a monomorphism. Thus f is monic, and so the embedding $f : M \to \mathrm{E}(R/P)$ guarantees that $\mathrm{rad}_{E(R/P} \leq \mathrm{rad}_{E(M)} = \sigma$, which completes the proof. $\square$

For a prime ideal P of a commutative ring R, the torsion radical defined by $\mathrm{E}(R/P)$ coincides with that defined by the complement of P. In general, for an ideal I of a ring R, let $C(I) = \{c \in R \mid cr \in I \text{ or } rc \in I \text{ implies } r \in I\}$. Thus $C(I)$ denotes the set of elements of R whose images are regular in R/I. The module ${}_RM$ is said to be $C(I)$-torsion if for each $m \in M$ there exists $c \in C(I)$ such that $cm = 0$, and then a torsion radical $\mathrm{rad}_{C(I)}$ can be defined by letting $\mathrm{rad}_{C(I)}(N)$ be the sum in N of all $C(I)$-torsion submodules, for all $N \in R\text{–}Mod$. That is, $\mathrm{rad}_{C(I)}(N) = \{x \in N \mid Rx \text{ is } C(I)\text{-torsion}\}$.

To show that $\mathrm{rad}_{C(I)}$ is in fact a torsion radical, we note that if $\mathrm{rad}_{C(I)}(M/\mathrm{rad}_{C(I)}(M)) \neq 0$, then there exists $m \in M$ such that $m \notin \mathrm{rad}_{C(I)}(M)$, but given $r \in R$ there exists $c \in C(I)$ such that

$crm \in \mathrm{rad}_{C(I)}(M)$. Then there exists $c' \in C(I)$ such that $c'crm = 0$, and since $c'c \in C(I)$, this implies that $m \in \mathrm{rad}_{C(I)}(M)$, a contradiction. The remaining conditions follow easily from the definition since a submodule of a $C(I)$-torsion module is again $C(I)$-torsion.

Observe that ${}_RM$ is $\mathrm{rad}_{C(I)}$-torsionfree (or simply $C(I)$-torsionfree) if for each $0 \neq m \in M$ there exists $r \in R$ such that $\mathrm{Ann}(rm) \cap C(I) = \emptyset$, and so in particular R/I is $C(I)$-torsionfree and hence $\mathrm{rad}_{E(R/I)} \geq \mathrm{rad}_{C(I)}$. If R satisfies the left Ore condition with respect to $C(I)$, that is, if for each $c \in C(I)$ and $r \in R$ there exist $c' \in C(I)$ and $r' \in R$ such that $r'c = c'r$, then for all $M \in R\text{–}Mod$, $\mathrm{rad}_{C(I)}(M) = \{m \in M \mid cm = 0 \text{ for some } c \in C(I)\}$. Note that this condition holds iff R/Rc is $C(I)$-torsion for all $c \in C(I)$, or equivalently, when I is a semiprime Goldie ideal, iff the elements of $C(I)$ are not zero divisors on $\mathrm{E}(R/I)$. If R satisfies the left Ore condition with respect to $C(0)$, then the corresponding ring of quotients is the left classical ring of quotients of R, denoted by $Q_{\mathrm{cl}}(R)$. If R is a semiprime, left Goldie ring, then $Q_{\mathrm{cl}}(R) = Q_{\max}(R)$ is semisimple Artinian by Goldie's theorem. The following proposition is due to Jategaonkar [14].

Proposition 3.2 *If I is a semiprime Goldie ideal of R, then* $\mathrm{rad}_{E(R/I)} = \mathrm{rad}_{C(I)}$.

Proof. The proof requires only that every dense left ideal of R/I contains a regular element, which includes the case in which R/I is a semiprime left Goldie ring. Since $\mathrm{rad}_{C(I)} \leq \mathrm{rad}_{E(R/I)}$, suppose that they are not equal. Then there exists a nonzero module ${}_RM$ which is $C(I)$-torsionfree but $\mathrm{rad}_{E(R/I)}$-torsion. By the definition of $\mathrm{rad}_{C(I)}$ there exists $m \in M$ such that $A \cap C(I) = \emptyset$ for $A = \mathrm{Ann}(m)$, and then $A + I/I$ does not contain a regular element of R/I since $(A + I) \cap C(I) = \emptyset$. Since M is $\mathrm{rad}_{E(R/I)}$-torsion, A and (hence) $A + I$ are $\mathrm{rad}_{E(R/I)}$-dense in R, and therefore $A + I/I$ is dense in R/I, contradicting the hypothesis. □

If I is a semiprime Goldie ideal, and $\{P_i\}_{i=1}^n$ are the minimal prime ideals of R/I, then as in Proposition 1.7, $Q_{\mathrm{cl}}(R/I) \simeq \prod_{i=1}^n Q_{\mathrm{cl}}(R/P_i)$. If $c \in C(I)$, then c is invertible in $Q_{\mathrm{cl}}(R/I)$, so it must be regular modulo P_i, for each i. The converse can be shown easily, so it follows that $C(I) = \cap_{i=1}^n C(P_i)$. Furthermore, $\mathrm{E}({}_{R/I}R/I) \simeq \oplus_{i=1}^n \mathrm{E}({}_{R/I}R/P_i)$, and so $\mathrm{E}(R/I) \simeq \oplus_{i=1}^n \mathrm{E}(R/P_i)$, which implies that $\mathrm{rad}_{E(R/I)} = \cap_{i=1}^n \mathrm{rad}_{E(R/P_i)}$, as was shown by Jategaonkar [14].

Proposition 3.3 *Let P and P' be prime Goldie ideals of R. Then $P \subseteq P'$ $\iff$ $\mathrm{rad}_{E(R/P} \geq \mathrm{rad}_{E(P')}$.*

Proof. $\Leftarrow$). This holds when P' is any prime ideal, by an argument used in both Theorem 1.6 and Proposition 2.6.

$\Rightarrow$). This requires only that P is a prime ideal of R for which every nonzero injective R/P-module is faithful, a condition which holds for any strongly prime ideal. Then $P = \mathrm{Ann}(\mathrm{E}({}_{R/P}R/P'))$, which shows that P is $\mathrm{rad}_{E(R/P')}$-closed, since $\mathrm{E}({}_{R/P}R/P') \subseteq \mathrm{E}(R/P')$, and so $\mathrm{rad}_{E(R/P)} \geq \mathrm{rad}_{E(R/P')}$. $\square$

For any ideal I of R, the exact sequence $0 \to I \to R \to R/I \to 0$ of R-modules gives rise to the exact sequence $0 \to I_\sigma \to R_\sigma \to (R/I)_\sigma$ of R_σ-modules, since the quotient functor determined by the torsion radical σ is left exact when viewed as a functor from R–*Mod* into R_σ–*Mod*. We will identify R_σ/I_σ with the corresponding submodule of $(R/I)_\sigma$. We note that although I is an ideal of R, I_σ may be only a left ideal of R_σ.

Proposition 3.4 *Let I be an ideal of R and let $\sigma = \mathrm{rad}_{E(R/I)}$.*

- **(a)** $Q_{\max}(R/I) = \{x \in (R/I)_\sigma \mid I_\sigma x = 0\}$.
- **(b)** $R_\sigma/I_\sigma \subseteq Q_{\max}(R/I) \Leftrightarrow I_\sigma$ *is an ideal of* R_σ.
- **(c)** *If I_σ is an ideal of R_σ, then R_σ/I_σ is a subring of $Q_{\max}(R/I)$.*
- **(d)** *If I_σ is an ideal of R_σ and σ is perfect, then $R_\sigma/I_\sigma = Q_{\max}(R/I)$.*

Proof. (a) $Q_{\max}(R/I)$ is defined by the R/I-injective envelope of R/I given by $\mathrm{E}({}_{R/I}R/I) = \{x \in \mathrm{E}(R/I) \mid Ix = 0\}$. Since $\mathrm{E}({}_{R/I}R/I)$ is a fully invariant submodule of $\mathrm{E}(R/I)$ and $Q_{\max}(R) = \{x \in \mathrm{E}({}_{R/I}R/I) \mid f(x) = 0$ for all $f \in \mathrm{End}(\mathrm{E}({}_{R/I}R/I))$ such that $f(R/I) = 0\}$, we must have $Q_{\max}(R) = \mathrm{E}({}_{R/I}R/I) \cap (R/I)_\sigma$. The result follows from the fact that for any element m of a σ-torsionfree left R_σ-module, $Im = 0$ iff $I_\sigma m = 0$.

(b) I_σ is an ideal of R_σ iff $I_\sigma R_\sigma \subseteq I_\sigma$, that is, iff $I_\sigma(R_\sigma/I_\sigma) = 0$, which occurs by part (a) iff $R_\sigma/I_\sigma \subseteq Q_{\max}(R/I)$.

(c) If I_σ is an ideal of R_σ, then the R-homomorphism $\pi : R_\sigma \to (R/I)_\sigma$ induced by $R \to R/I \to 0$ maps R_σ into $Q_{\max}(R/I)$, and $\pi(1) = 1$. Since for $p, q \in R_\sigma$, $\pi(pq) = \pi\rho_q(p)$ and $\pi(p)\pi(q) = \rho_{\pi(q)}\pi(p)$, where ρ_x defines multiplication by elements of R_σ, to show that π is a ring homomorphism it suffices to show that $\pi\rho_q = \rho_{\pi(q)}\pi$. These map into $\mathrm{E}(R/I)$ and so they will be equal if they agree on $R/\mathrm{rad}_\sigma(R)$. This holds since $\pi\rho_q(1) = \pi(q) = \rho_{\pi(q)}(1) = \rho_{\pi(q)}\pi(1)$.

(d) If σ is perfect, then $Q_\sigma : R\text{–}Mod \to R_\sigma\text{–}Mod$ is exact, so $R_\sigma/I_\sigma = (R/I)_\sigma$, and this implies that $R_\sigma/I_\sigma = Q_{\max}(R/I)$. □

We note that Proposition 3.4 (a) generalizes Proposition 3.3 of [12], and as in Proposition 3.4 of [12], we can show that for the idealizer $\mathcal{I}(I_\sigma) = \{q \in R_\sigma \mid I_\sigma q \subseteq I_\sigma\}$, we have $\mathcal{I}(I_\sigma)/I_\sigma = Q_{\max}(R/I) \cap (R_\sigma/I_\sigma)$. Parts (b) and (c) of Proposition 3.4 extend Lemma 2.3 of [16] and part of Theorem 3.6 of [12]. The next proposition extends parts of Theorems 3.6 and 3.7 of [12].

Proposition 3.5 *Let P be a prime ideal of R, with $\sigma = \mathrm{rad}_{E(R/P)}$.*

- **(a)** *P_σ is an ideal of R_σ $\Leftrightarrow$ R_σ/P_σ is a prime R-module.*
- **(b)** *The following conditions are equivalent:*
  (1) P_σ is an ideal of R_σ and $(R/P)_\sigma$ is a prime R_σ-module;
  (2) $(R/P)_\sigma = Q_{\max}(R/P)$;
  (3) $(R/P)_\sigma$ is a prime R_σ-module.

Proof. (a) If P_σ is an ideal of R_σ, then $R_\sigma/P_\sigma \subseteq \mathrm{E}({}_{R/P}R/P)$, and the latter is prime. Conversely, if R_σ/P_σ is prime, then $P = \mathrm{Ann}(R/P)$ implies $P = \mathrm{Ann}(R_\sigma/P_\sigma)$, so $P_\sigma R_\sigma \subseteq P_\sigma$.

(b) (1) $\Rightarrow$ (2). If P_σ is an ideal of R_σ, then $P_\sigma = \mathrm{Ann}(R_\sigma/P_\sigma)$, so if $(R/P)_\sigma$ is a prime R_σ-module, then $P_\sigma = \mathrm{Ann}((R/P)_\sigma)$ and $(R/P)_\sigma = Q_{\max}(R/P)$ by Proposition 3.4.

(2) $\Rightarrow$ (3). $Q_{\max}(R/P)$ is a prime R/P-module, so it is prime as an R-module.

(3) $\Rightarrow$ (1). If $(R/P)_\sigma$ is a prime R-module, then P_σ is an ideal by part (a) since $R_\sigma/P_\sigma \subseteq (R/P)_\sigma$. Thus P_σ is a prime ideal, and $(R/P)_\sigma$ is a prime R_σ-module since it is contained in the R_σ/P_σ-injective envelope of R_σ/P_σ. □

Theorem 3.6 ([7]) *Let I be a semiprime Goldie ideal of the ring R. The following conditions are equivalent for $\sigma = \mathrm{rad}_{E(R/I)}$:*

(1) *$I_\sigma = \mathrm{J}(R_\sigma)$ and R_σ is a semi-local ring;*

(2) *I_σ is an ideal of R_σ and σ is perfect;*

(3) *R satisfies the left Ore condition with respect to $C(I)$, and for each $c \in C(I)$ there exists $r \in R$ such that $rc \in C(I)$ and such that for each $a \in R$ with $arc = 0$ there exists $c' \in C(I)$ with $c'ar = 0$.*

Proof. (1) $\Rightarrow$ (2). If $I_\sigma = \mathrm{J}(R_\sigma)$, then $R_\sigma/\mathrm{J}(R_\sigma) = R_\sigma/I_\sigma \subseteq \mathrm{E}(R/I)$, and if $R_\sigma/\mathrm{J}(R_\sigma)$ is semisimple Artinian, then $\mathrm{E}(R/I)$ contains an isomorphic copy of each simple left R_σ-module, so it is a cogenerator. It follows from [4, Proposition 3.7] that σ is perfect.

(2) $\Rightarrow$ (1). If I_σ is an ideal and σ is perfect, then by Proposition 3.4, $R_\sigma/I_\sigma = Q_{\mathrm{cl}}(R/I)$ is semisimple Artinian since R/I is a semiprime Goldie ring, and thus $I_\sigma \supseteq \mathrm{J}(R_\sigma)$. On the other hand, since σ is perfect, $\mathrm{E}(R/I) = \mathrm{E}(R_\sigma/I_\sigma)$ must contain an isomorphic copy of each simple R_σ-module. Therefore I_σ annihilates each simple R_σ-module, and $I_\sigma \subseteq \mathrm{J}(R_\sigma)$.

(1) and (2) $\Rightarrow$ (3). Since R/I is a semiprime left Goldie ring,

$\sigma = \mathrm{rad}_{C(I)}$ and $Rc + I/I$ is σ-dense in R/I for all $c \in C(I)$, so $Rc + I$ is σ-dense for all $c \in C(I)$. Therefore $R_\sigma c + I_\sigma = R_\sigma(Rc + I) = R_\sigma$ since σ is perfect, and since $I_\sigma = \mathrm{J}(R_\sigma)$ by condition 1, $R_\sigma = R_\sigma c = R_\sigma Rc$, which shows that Rc is σ-dense. Thus R satisfies the left Ore condition with respect to $C(I)$. The second part of condition 3 holds by [21, Proposition 15.3] since σ is perfect.

(3) $\Rightarrow$ (2). By [21, Proposition 15.3], σ is perfect. Since the left Ore condition is satisfied, R_σ can be constructed as a set of ordered pairs of elements, subject to the usual equivalence relation, and a direct computation shows that $I_\sigma = R_\sigma I$ is an ideal of R_σ. □

In Theorem 3.6 the assumption that I is a semiprime Goldie ideal can be removed. In [5] it was shown that for any ideal I of R, $I_\sigma = \mathrm{J}(R_\sigma)$ and R_σ is local iff I_σ is a ideal of R_σ, σ is perfect, and the ring R/I has finite uniform dimension and zero singular ideal.

If R is left Noetherian, then condition 3 can be simplified to give Theorem 5.6 of [16]. As Lambek and Michler showed in [16], if R is left Noetherian, I is a semiprime ideal of R, and $ac = 0$ for $a \in R$, $c \in C(I)$, then we can choose a positive integer n such that $\ell(c^n)$ is maximal. If the left Ore condition is satisfied with respect to $C(I)$, then there exist $a' \in R$ and $c' \in C(I)$ such that $a'c^n = c'a$. Since $\ell(c^n)$ is maximal, $\ell(c^{n+1}) = \ell(c^n)$, and then $a'c^{n+1} = c'ac = 0$ shows that $a'c^n = 0$ and hence $c'a = 0$. Thus if R is left Noetherian, then condition 3 of Theorem 3.6 is equivalent to the condition that R satisfies the left Ore condition with respect to $C(I)$. An additional equivalent condition given by Lambek and Michler is that I/Ic is σ-torsion for each $c \in C(I)$.

In the following example from [15], the ring R is left Noetherian but does not satisfy the left Ore condition with respect to $C(P)$ for its prime ideal P, even though the torsion radical defined by $\mathrm{E}(R/P)$ is perfect. Let D be a commutative local Noetherian domain such that $J = \mathrm{J}(D)$ is principal. Let R be the ring of matrices $\begin{pmatrix} D & J \\ D & D \end{pmatrix}$, and let

P be the prime ideal $\begin{pmatrix} J & J \\ D & D \end{pmatrix}$. The ring of quotients defined by P is just $\begin{pmatrix} D & D \\ D & D \end{pmatrix}$.

On the other hand, any ideal of the full ring of $n \times n$ matrices over a commutative ring satisfies condition 3. This can be shown by using the fact that a matrix over a commutative ring is regular iff its determinant is regular.

Corollary 3.7 *If the conditions of Theorem 3.6 hold, then (with the notation of Theorem 3.6) the localization M_σ of any R/I-module is an R_σ/I_σ-module, and hence it is a direct sum of simple R_σ-modules.*

Proof. If M is an R/I-module, then $IM = 0$ and hence $IM'' = 0$ for $M'' = M/\text{rad}_\sigma(M)$. If $x \in M_\sigma$, then $Dx \subseteq M''$ for some σ-dense left ideal D, and so $IDx = 0$. But then $I_\sigma Dx = 0$, and so since σ is perfect and I_σ is an ideal we must have $I_\sigma x = I_\sigma R_\sigma x = I_\sigma R_\sigma Dx = I_\sigma Dx = 0$. □

The following theorem extends Small's theorem [19] characterizing rings whose classical ring of quotients is left Artinian. The Noetherian case was given by Lambek and Michler, although the proof below is taken from [7], in which the theorem was proved in its full generality. The assumption that R is left Noetherian can be dropped if the following conditions are added: (iii) R/I and R/K are left Goldie rings and (iv) $R/\{r \in R \mid I^m r \subseteq K\}$ has finite uniform dimension for all integers $1 \leq m \leq k$, where $I^k \subseteq K$ but $I^{k-1} \not\subseteq K$.

Theorem 3.8 ([16]) *Let I be a semiprime ideal of a left Noetherian ring R, and let $\sigma = \text{rad}_{E(R/I)}$, with $K = \text{rad}_\sigma(R)$. Then R_σ is a left Artinian classical ring of fractions of R with respect to $C(I) \iff$ (i) $I^k \subseteq K$ for some positive integer k and (ii) for each $r \in R$ and $c \in C(I)$, $rc = 0$ implies there exists $c' \in C(I)$ such that $c'r = 0$.*

Proof. Assume that R_σ is a left Artinian classical ring of left fractions with respect to $C(I)$. Since R_σ is a partial classical ring of left quotients of R/K, it follows that $R_\sigma = Q_{\text{cl}}(R/K)$ since R_σ is left

Artinian. By [21, Proposition 15.7], condition (ii) holds, and thus the conditions of Theorem 3.6 are satisfied, since R must satisfy the left Ore condition with respect to $C(I)$. But then $R_\sigma I = I_\sigma = \mathrm{J}(R_\sigma)$ is nilpotent, and so $I^k \subseteq K$ for some positive integer k. This shows that (i) holds and moreover that I/K is the prime radical of R/K.

Conversely, $\mathrm{rad}_{E(R/I)} = \mathrm{rad}_{C(I)}$ by Proposition 3.2, and so each element of $C(I)$ is left regular module $K = \mathrm{rad}_{C(I)}(R)$. To show this, let $r \in R$ and $c \in C(I)$ and assume that $rc \in K$. Then for each $s \in R$, there exists $c' \in C(I)$ such that $c'src = 0$, so by condition (ii) there exists $c'' \in C(I)$ such that $c''csr = 0$, and so $r \in K$.

To show that R satisfies the left Ore condition with respect to $C(I)$ it suffices to show that R/Rc is $C(I)$-torsion for all $c \in C(I)$. Let $I_0 = K$ and $I_m = \{r \in R \mid I^m r \subseteq K\}$, so that $I_k = R$ if $I^k \subseteq K$ but $I^{k+1} \nsubseteq K$. If $rc \in I_m$ for $r \in R$ and $c \in C(I)$, then $I^m rc \subseteq K$, and since c is left regular module K, $I^m r \subseteq K$ and hence $r \in I_m$. Thus each element of $C(I)$ is left regular modulo I_m and so $I_m \cap I_{m+1}c = I_m c$ for $0 \leq m \leq k$.

Now for $0 \leq m \leq k$, $I_{m+1} \supseteq I_m + I_{m+1}c \supseteq I_{m+1}c$ and $(I_m + I_{m+1}c)/I_{m+1}c \simeq I_m/I_m \cap I_{m+1}c = I_m/I_m c$. Thus to show that $I_{m+1}/I_{m+1}c$ is σ-torsion it suffices to show that both $I_{m+1}/I_m + I_{m+1}c$ and $I_m/I_m c$ are σ-torsion, so that R/Rc is σ-torsion if $I_{m+1}/I_m + I_{m+1}c$ is σ-torsion for $0 \leq m \leq k$ ($I_0/I_0 c = K/Kc$ is a factor of $K = \mathrm{rad}_\sigma(R)$ and so it must be σ-torsion).

Let $x \in I_{m+1}$, and let $A = \{r \mid rx \in I_m + I_{m+1}c\}$. If A/I is essential in R/I, then since R/I is a semiprime Goldie ring, there exists $c' \in C(I)$ such that $c' \in A$ and $I_{m+1}/I_m + I_{m+1}c$ will be σ-torsion. Accordingly, to show that A/I is essential in R/I, let $r \in R$ and $r \notin A$. Then $rx \notin I_m$, since $rx \in I_m$ implies $r \in A$, and so $0 \neq rx + I_m$ and $I_{m+1}c + I_m/I_m$ is essential in I_{m+1}/I_m since c is left regular modulo I_m and R/I_m is by assumption finite dimensional. Thus there exists $s \in R$ such that $srx \in I_{m+1}c + I_m$ but $srx \notin I_m$. Now $sr \in I$ implies

$I^m srx \subseteq I^{m+1}x = \subseteq K$, a contradiction. Thus $sr \notin I$, and A/I is essential in R/I.

Since it has now been established that R satisfies the left Ore condition with respect to $C(I)$, if $cr \in K$ for $r \in R$, $c \in C(I)$, then there exists $c' \in C(I)$ such that $c'cr = 0$, so $r \in K$. Thus $C(I) \subseteq C(K)$ and since $I^k \subseteq K$, I/K is the prime radical of R/K, and R/K satisfies the hypothesis of Small's theorem. It follows that $C(I) = C(K)$ and thus $R_\sigma = Q_{\mathrm{cl}}(R/K)$ is left Artinian. $\square$

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