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FINITELY ANNIHILATED MODULES AND ORDERS IN ARTINIAN RINGS

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A module will be called finitely annihilated if its annihilator is the annihilator of a finite subset. A ring is Artinian if and only if every module over it is finitely annihilated. This suggests that finitely annihilated modules should be useful in studying orders in Artinian rings, since if S is a flat epimorphic image of R, then an S-module is finitely annihilated if and only if it is finitely annihilated as an R-module.

We show in Theorem 2.4 that a ring R with prime radical N is an order in an Artinian ring if and only if (a) $\mathrm{E}(R/N)$ is faithful; (b) every submodule of $\mathrm{E}(R/N)$ is finitely annihilated; (c) $\mathrm{E}(R/N)$ defines a perfect torsion radical σ; (d) $N \cdot (R/N)_\sigma = 0$. This weakens the hypothesis of Tachikawa's characterization of orders in Artinian rings [18, Theorem 3.1], by replacing the descending chain condition on annihilators of subsets with condition (b). We show that an indecomposable, injective module W over a ring with Krull dimension defines a maximal torsion radical if and only if every submodule of W is finitely annihilated, which implies that condition (b) holds for any ring with Krull dimension. We also show that if R has Krull dimension, then $\mathrm{E}(R/N)$ is faithful if and only if every associated prime torsion radical of R is maximal.

If R is an order in an Artinian ring, then R has finite Goldie dimension and every ideal of R is finitely annihilated. For semiprime rings this condition characterizes orders in (semisimple) Artinian rings. The condition is Morita invariant, so it may prove to be a useful variant of Goldie's conditions, since it is as yet unknown whether even the ring of $n \times n$ matrices over a Goldie ring must be Goldie.

In Section 1 we give some elementary properties of finitely annihilated modules. Section 2 contains the results mentioned above on orders in Artinian rings. Any finitely generated module over a commutative ring is finitely annihilated, so in Section 3 we investigate finitely generated, finitely annihilated modules and their connections with linear topologies. In Section 4 we consider rings whose ideals are finitely annihilated, and list several questions.

All rings will be associative rings with identity, and for a ring R the category of left R-modules will be denoted by R–Mod. A subfunctor σ of the identity 1 on R–Mod is called a torsion preradical if $\sigma(N) = N \cap \sigma(M)$ for all R-submodules N of M, and a torsion radical if, in addition, $\sigma(M/\sigma(M)) = 0$ for all R-modules M. The module M is called σ-torsion if $\sigma(M) = M$ and σ-torsionfree if $\sigma(M) = 0$, and the submodule N of M is called σ-dense in M if M/N is σ-torsion and σ-closed in M if M/N is σ-torsionfree. A torsion radical σ is called maximal if it is proper ($\sigma \neq 1$) and maximal with respect to the relation $\leq$, where $\tau \leq \sigma$ for torsion radicals τ and σ if $\tau(M) \subseteq \sigma(M)$ for all $M \in R\text{–}Mod$. The torsion radical cogenerated by an injective R-module W is denoted by rad_W, where $\text{rad}_W(M) = \cap_{f \in \text{Hom}(M,W)} \ker(f)$. If $\text{E}(M)$ is the injective envelope of M, then $\text{rad}_{E(M)}$ is the largest torsion radical for which M is torsionfree. As a general reference for torsion theoretic matters we refer the reader to Stenström [17].

A module M is said to have finite Goldie dimension if there are no infinite direct sums of submodules in M. A module M is quasi-injective if every homomorphism from a submodule of M into M can be lifted to an endomorphism of M. A fully invariant submodule of a module is a submodule which is mapped into itself by every endomorphism of the module. Thus a module M is quasi-injective if and only if M is a fully invariant submodule of $\text{E}(M)$. The quasi-injective hull of M, denoted by $\overline{M}$, is $\sum\{f(M) \mid f \in \text{End}_R(\text{E}(M))\}$. The singular submodule of M is $\text{Z}(M) = \{m \in M \mid \text{Ann}(m) \text{ is essential in } R\}$.

1 Finitely annihilated modules

DEFINITION 1.1 *The module ${}_RM$ is said to be finitely annihilated if there exist $m_1, \ldots, m_k \in M$ such that* $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \ldots, m_k)$.

Observe that ${}_RM$ is finitely annihilated if and only if there exists an embedding $0 \to R/\mathrm{Ann}(M) \to M^k$ for a direct sum M^k of copies of M. If ${}_RM$ is finitely annihilated and $M \subseteq {}_RN$ with $\mathrm{Ann}(M) = \mathrm{Ann}(N)$, then N is finitely annihilated. This shows that if M is finitely annihilated, then so is any direct sum or direct product of copies of M. Furthermore, for any module M, the existence of an embedding $0 \to R/\mathrm{Ann}(M) \to \prod_{\alpha \in I} M_\alpha$ for the set $I = M$, with $M_\alpha = M$, shows that $\prod_{\alpha \in I} M$ is finitely annihilated. Any submodule of a module with the descending chain condition on annihilators must be finitely annihilated. As a partial converse, we have the following proposition.

PROPOSITION 1.2 *If every submodule of ${}_RM$ is finitely annihilated, then the descending chain condition holds for annihilators of submodules of M.*

Proof. Let $\mathrm{Ann}(M_1) \supseteq \mathrm{Ann}(M_2) \supseteq \cdots$ be a descending chain of annihilators of submodules of M. Let $N_m = \sum_{i=1}^m M_i$ and observe that $\mathrm{Ann}(N_m) = \mathrm{Ann}(M_m)$. By hypothesis there exist $x_1, \ldots, x_t \in \sum_{i=1}^\infty M_i$ such that $\mathrm{Ann}(\sum_{i=1}^\infty M_i) = \mathrm{Ann}(x_1, \ldots, x_t)$. Furthermore, there exists k such that $\{x_1, \ldots, x_t\} \subseteq N_j$ for all $j \geq k$. Thus for $j \geq k$, $\mathrm{Ann}(M_j) = \mathrm{Ann}(N_j) = \mathrm{Ann}(N_k) = \mathrm{Ann}(M_k)$. □

We remark that a slight modification of the above proof shows that the descending chain condition holds for annihilators of submodules of any Noetherian module. On the other hand, the example of Small [16] cited in [6] shows that not every (faithful) left ideal of a left Noetherian ring is finitely annihilated.

LEMMA 1.3 (*a*) *Let ${}_RM$ and ${}_RN$ be modules such that (i) M can be embedded in a direct sum of copies of N and (ii) N is a homomorphic image of a direct sum of copies of M. Then M is finitely annihilated if and only if N is finitely annihilated.*

(*b*) *Let M and N be modules such that M can be embedded in a direct product of copies of N. If every submodule of N is finitely annihilated, then the same condition holds for M.*

(*c*) *If every fully invariant submodule of M is finitely annihilated, then every submodule of M is finitely annihilated.*

(*d*) *If $\{N_i\}_{i=1,\dots,n}$ is a collection of modules for which every submodule is finitely annihilated, then the same condition holds for $\oplus_{i=1}^n N_i$.*

Proof. (a) Let $f : M \to \oplus_{\alpha\in I} N_\alpha$ be the given monomorphism and let $f : \oplus_{\alpha\in J} M_\alpha \to N$ be the given epimorphism. It is clear that $\mathrm{Ann}(M) = \mathrm{Ann}(N)$. If M is finitely annihilated, say $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \dots, m_k)$, then $\{f_\alpha(m_i)\}_{\alpha\in I},\ 1 \le i \le k$ has only finitely many nonzero elements. Here f_α denotes the αth component of f. If $rf_\alpha(m_i) = 0$ for all α and i, then $rf(m_i) = 0$ for all i, which implies that $rm_i = 0$ for all i. Thus $r \in \mathrm{Ann}(M) = \mathrm{Ann}(N)$. If N is finitely annihilated, say $\mathrm{Ann}(N) =$

$\text{Ann}(n_1, \ldots, n_k)$, then letting g_α denote the αth component of g and picking a representative in $g_\alpha^{-1}(n_i)$ we again have only finitely many nonzero elements and $rg_\alpha^{-1}(n_i) = 0$ for all α and i implies that $rn_i = 0$ for all i, and so $r \in \text{Ann}(N) = \text{Ann}(M)$.

(b) If M' is a submodule of M, let $N' \subseteq N$ be the sum in N of all homomorphic images of M'. Then since M' can be embedded in a direct product of copies of N', we have $\text{Ann}(M') = \text{Ann}(N')$, and as in the proof of part (a) above, M' must be finitely annihilated since N' is a homomorphic image of a direct sum of copies of M'.

(c) If M' is a submodule of M, let N be the sum in M of all homomorphic images of M'. If every fully invariant submodule of M is finitely annihilated, then N must be finitely annihilated, and so by part (a), M' must be finitely annihilated.

(d) By part (c) it is sufficient to consider only fully invariant submodules of $\oplus_{i=1}^n N_i$, which must be of the form $\oplus_{i=1}^n N_j'$ for submodules $N_i' \subseteq N_i$. It is clear that by assumption every submodule of this form is finitely annihilated. □

In [1] it was shown that every faithful left R-module is finitely annihilated if and only if R is essentially left Artinian (that is, R contains an essential Artinian left ideal, or equivalently, the left socle of R is essential and finitely generated). Thus every left R module is finitely annihilated if and only if every factor ring of R is essentially left Artinian, which occurs if and only if R is left Artinian [1, Proposition 5]. The next proposition follows from the above remarks and Lemma 1.3 (b). As the following propositions will show, it is essentially the same as a theorem of Faith [9, Theorem 17A].

PROPOSITION 1.4 *The following conditions are equivalent.*

(1) *R is left Artinian.*

(2) *Every left R-module is finitely annihilated.*

(2) *R–Mod has a cogenerator W such that every submodule of W is finitely annihilated.*

PROPOSITION 1.5 *Let $\overline{M}$ denote the quasi-injective hull of ${}_RM$. The following conditions are equivalent.*

(1) *M is finitely annihilated.*

(2) *$\overline{M}$ is finitely annihilated.*

(2) *$\overline{M}$ is finitely generated as a module over its R-endomorphism ring.*

Proof. (1) $\Leftrightarrow$ (2) This follows from Lemma 1.3 (a), since $\overline{M}$ is the sum in $\mathrm{E}(M)$ of all homomorphic images of M.

(2) $\Leftrightarrow$ (3) This follows from [2, Proposition 1.6], since $\overline{M}$ is faithful as an $R/\operatorname{Ann}(\overline{M})$-module, and if $\overline{M}$ is finitely annihilated, then it is injective over $R/\operatorname{Ann}(\overline{M})$. $\square$

PROPOSITION 1.6 *The following conditions are equivalent for a quasi-injective module ${}_RW$.*

(1) *Every submodule of W is finitely annihilated.*

(2) *Every fully invariant submodule of* W *is finitely generated as a right* $\mathrm{End}_R(W)$*-module.*

Proof. Since W is quasi-injective, every fully invariant submodule is quasi-injective, and endomorphisms of submodules can be lifted to $\mathrm{End}_R(W)$, so (1) $\Leftrightarrow$ (2) by Lemma 1.3 (c) and Proposition 1.5. $\square$

A module ${}_RM$ is called prime if $\mathrm{Ann}(M') = \mathrm{Ann}(M)$ for all nonzero submodules $M' \subseteq M$; M is called strongly prime if it is prime and for each nonzero submodule $M' \subseteq M$ and each $y \in M$ there exist elements $x_1, \ldots, x_k \in M'$ such that $\cap_{i=1}^n \mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(y)$. The following conditions are equivalent (see [6]): (1) M is strongly prime; (2) for each $0 \neq x \in M$ and each $y \in M$ there exist elements $r_1, \ldots, r_k \in R$ such that $\cap_{i=1}^n \mathrm{Ann}(rx_i) \subseteq \mathrm{Ann}(y)$; (3) M is contained in every nonzero fully invariant submodule of $\mathrm{E}(M)$; (4) for any torsion preradical τ of R–Mod, either $\tau(M) = M$ or $\tau(M) = 0$. The ring R is called (left) strongly prime if ${}_RR$ is strongly prime; it is called (left) strongly semiprime if R is semiprime and every faithful left ideal of R is finitely annihilated. If R is strongly semiprime, then $\mathrm{Z}(R) = 0$.

PROPOSITION 1.7 *The following conditions are equivalent for* ${}_RM$*.*

(1) M *is strongly prime and finitely annihilated.*

(2) M *is prime and every submodule of* M *is finitely annihilated.*

(3) *M is prime and every finitely generated submodule of M is finitely annihilated.*

Proof. (1) $\Rightarrow$ (2) If M is strongly prime, then $\mathrm{Ann}(M') = \mathrm{Ann}(M)$ for any nonzero submodule $M' \subseteq M$. Thus it is sufficient to show that every cyclic submodule is finitely annihilated. If M is finitely annihilated, then $\mathrm{Ann}(y_1, \ldots, y_m) = \mathrm{Ann}(M)$ for some $y_1, \ldots, y_m \in M$. By assumption, if $0 \neq x \in M$, then there exist $r_{1j}, r_{2j}, \ldots, r_{kj} \in R$ such that $\cap_{i=1}^{k} \mathrm{Ann}(r_{ij}x) \subseteq \mathrm{Ann}(y_j)$, for $1 \leq j \leq m$. Thus $\cap_{i,j} \mathrm{Ann}(r_{ij}x) \subseteq \cap_{j=1}^{m} \mathrm{Ann}(y_j) = \mathrm{Ann}(M) = \mathrm{Ann}(Rx)$, and so Rx is finitely annihilated.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Assume that M is prime and that every finitely generated submodule of M is finitely annihilated. If $0 \neq x \in M$ and $y \in M$, then Rx is finitely annihilated, so there exist elements $r_1, \ldots, r_k \in R$ such that $\cap_{i=1}^{k} \mathrm{Ann}(r_i x) = \mathrm{Ann}(Rx) = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(y)$. Thus M is strongly prime and finitely annihilated. □

COROLLARY 1.8 *If R is strongly semiprime, then every submodule of* $\mathrm{E}(R)$ *is finitely annihilated.*

Proof. If R is strongly semiprime, then R is an essential subdirect product $R \to \prod_{i=1}^{n} R/P_i$ of strongly prime rings R/P_i. (See Handelman [13].) Thus $\mathrm{E}(R) = \oplus_{i=1}^{n} \mathrm{E}(R/P_i)$, and by the remarks following Theorem 2 of [5], $P_i = \mathrm{Ann}(\mathrm{E}(R/P_i))$. Thus $\mathrm{E}(R/P_i)$ is the R/P_i-injective envelope of R/P_i, and it has no nontrivial invariant submodules since R/P_i is strongly prime, so $\mathrm{E}(R/P_i)$ is strongly prime. The corollary follows from Proposition 1.6 and Lemma 1.3 (d). □

2 Orders in Artinian rings

Let σ be a torsion radical. The quotient functor Q_σ determined by σ is defined for $M \in R\text{–}Mod$ as $\{x \in \mathrm{E}(M/\sigma(M)) \mid Dx \in M/\sigma(M)$ for some σ-dense left ideal $D\}$. The R-module $Q_\sigma(M)$, which will be denoted by M_σ, is a left module over the ring of quotients R_σ. If N is an ideal of R and $\sigma = \mathrm{rad}_{E(R/N)}$, then the localization R_σ at N will be denoted by R_N. The torsion radical σ is called perfect if one of the following equivalent conditions hold (see Stenström [17]): (1) σ is hereditary (i.e. $Q_\sigma : R\text{–}Mod \to R\text{–}Mod$ is exact) and every σ-dense left ideal of R contains a finitely generated σ-dense left ideal; (2) Q_σ is naturally isomorphic to $R_\sigma \otimes_R —$; (3) every R_σ-module is σ-torsionfree; (4) $R_\sigma \cdot D = R_\sigma$ for all σ-dense left ideals $D \subseteq R$.

If σ is perfect, then for any left ideal $A \subseteq R$, A_σ is naturally isomorphic to $R_\sigma \otimes_R A$, which may be identified with $R_\sigma \cdot A \subseteq R_\sigma$. In this case there is a one-to-one correspondence between σ-closed left ideals of R and left ideals of R_σ. Furthermore, the natural ring homomorphism $R \to R_\sigma$ is an epimorphism in the category of rings which makes R_σ into a flat right R-module. Conversely, if $R \to S$ is an epimorphism of rings such that S becomes a flat right R-module, then S is ring-isomorphic to a ring of the form R_σ for a perfect torsion radical σ.

PROPOSITION 2.1 *Let $\phi : R \to S$ be an epimorphism in the category of rings such that S_R is flat. An S-module M is finitely annihilated if and only if M is finitely annihilated as an R-module.*

Proof. If $\mathrm{Ann}_S(M) = \mathrm{Ann}_S(m_1, \ldots, m_k)$, then clearly $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(m_1, \ldots, m_k)$. Conversely, let $s \in S$ and suppose that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(m_1, \ldots, m_k)$. Then since S is a flat epimorphic image of R, there exist elements $b_1, \ldots, b_n \in R$ such that $\phi(b_i)s \in \phi(R)$ for $1 \leq i \leq n$ and $\sum_{i=1}^n c_i\phi(b_i) = 1$ for some elements $c_1, \ldots, c_n \in S$ [17, Chapter XI, Theorem 2.1]. Thus if $sm_j = 0$ for $1 \leq j \leq k$, then for any $i, 1 \leq i \leq n$, $\phi(b_i)sm_j = 0$ for all j implies that $\phi(b_i)sm = 0$ for all $m \in M$. Thus $sm = \sum_{i=1}^n c_i\phi(b_i)sm = 0$ for all $m \in M$, and $\mathrm{Ann}_S(m_1, \ldots, m_k) = \mathrm{Ann}_S(M)$. □

PROPOSITION 2.2 *Let σ be a perfect torsion radical of R–Mod cogenerated by the injective module ${}_RW$. The following conditions are equivalent.*

(1) *The ring of quotients R_σ is left Artinian.*

(2) *Every submodule of W is finitely annihilated.*

(3) *Every σ-torsionfree R-module is finitely annihilated.*

Proof. (1) ⇒ (2) If R_σ is left Artinian, then R satisfies the descending chain condition for σ-closed left ideals, since any such left ideal corresponds to a left ideal of R_σ. Since any left annihilator of W is a σ-closed left ideal, W has the descending chain condition on left annihilators, and so every submodule of W is finitely annihilated.

(2) $\Rightarrow$ (3) This follows from Lemma 1.3 (b) and the fact that W cogenerates σ.

(3) $\Rightarrow$ (1) Since R_σ is a flat epimorphic image of R and every left R_σ-module is σ-torsionfree, it follows from Proposition 2.1 that every left R_σ-module is finitely annihilated. Then R_σ is left Artinian by Proposition 1.4. □

The version of Goldie's Theorem presented in [6] states that a ring R is an order in a semisimple Artinian ring if and only if (a) every faithful left ideal of R is finitely annihilated; (b) R has finite left Goldie dimension; and (c) R is semiprime. We can reformulate this as follows.

PROPOSITION 2.3 *A ring R is an order in a semisimple Artinian ring if and only if (a) every submodule of $E(R)$ is finitely annihilated; (b) $\mathrm{rad}_{E(R)}$ is perfect; and (c) R is semiprime.*

Proof. Since R is semiprime, Corollary 1.8 implies that every submodule of $\mathrm{E}(R)$ is finitely annihilated if and only if every faithful left ideal of R is finitely annihilated. Since $\mathrm{Z}(R) = 0$ when R is strongly semiprime, [3, Proposition 3.9] implies that $\mathrm{rad}_{E(R)}$ is perfect if and only if R has finite left Goldie dimension. □

Our next result extends Proposition 2.3 to orders in arbitrary left Artinian rings. We can weaken the hypothesis of the main theorem of Tachikawa [18, Theorem 3.1] and simplify his proofs by making use of the general theory of localization at a semiprime ideal developed in [7]. Note that if R is semiprime, then conditions (2a) and (2d) automatically hold, and the theorem reduces to Proposition 2.3. As we will show in Corollary 2.6, condition (2b) holds for any ring with Krull dimension. Furthermore, for such rings condition (2a) is equivalent to the condition that every associated prime torsion radical of R is maximal, as we will show in Theorem 2.12. Thus the theorem reduces to the well-known result that a commutative, Noetherian ring is an order in an Artinian ring if and only if it has no embedded primes.

For an ideal I of R, let $\mathrm{C}(I) = \{c \in R \mid cr \in I \text{ or } rc \in I \Rightarrow r \in I\}$. Thus $\mathrm{C}(I)$ denotes the set of elements regular modulo I.

THEOREM 2.4 *Let R be a ring with prime radical N, and let R_N denote the localization of R at N. The following conditions are equivalent.*

(1) *R is a left order in a left Artinian ring.*

(2) *R is a ring such that*

(2a) $\mathrm{E}(R/N)$ *is faithful.*

(2b) Every submodule of $\mathrm{E}(R/N)$ *is finitely annihilated.*

(2c) $\mathrm{rad}_{E(R/N)}$ *is perfect.*

(2d) $N \cdot (R/N)_N = 0$.

(3) *There exists an injective module* $_RW$*, with* $\mathrm{rad}_W = \sigma$*, such that*

(3a) W *is faithful.*

(3b) Every submodule of W *is finitely annihilated.*

(3c) σ *is perfect.*

(3d) $R \cap \mathrm{J}(R_\sigma) = N$.

Proof. (1) $\Rightarrow$ (3) If R is a left order in the left Artinian ring $Q = Q_{cl}(R)$, let W be an injective cogenerator for Q–Mod. Then $_RW$ is an injective R-module, and as such it is faithful. Every submodule of W is finitely annihilated by Proposition 2.2 and σ is perfect since Q is a right flat epimorphic extension of R. Finally, $R \cap \mathrm{J}(R_\sigma) = R \cap \mathrm{J}(Q) = N$ by Robson [14], where $\mathrm{J}(R_\sigma)$ denotes the Jacobson radical of R_σ.

(3) $\Rightarrow$ (2) Since $R \cap \mathrm{J}(R_\sigma) = N$, $R/N \subseteq R_\sigma/\mathrm{J}(R_\sigma)$. If $q \in R_\sigma$, then $D = \{x \in R \mid xq \in R\}$ is a σ-dense left ideal of R, and so if $Dq \subseteq \mathrm{J}(R_\sigma)$, then $R_\sigma \cdot q = R_\sigma \cdot Dq \subseteq \mathrm{J}(R_\sigma)$ because σ is perfect and thus $R_\sigma \cdot D = R_\sigma$. If $q \in R_\sigma$ but $q \notin \mathrm{J}(R_\sigma)$, then there exists $r \in R$ with $rq \in R$ and $rq \notin \mathrm{J}(R_\sigma)$, and hence R/N is essential in $R_\sigma/\mathrm{J}(R_\sigma)$. Therefore $\mathrm{E}(R/N) = \mathrm{E}(R_\sigma/\mathrm{J}(R_\sigma))$ is a cogenerator for R_σ–Mod and $\mathrm{E}(R/N)$ must define σ, so $R_\sigma = R_N$. Since $R/N \subseteq R_N/\mathrm{J}(R_N)$, $N \cdot (R/N)_N \subseteq \mathrm{J}(R_N) \cdot (R/N)_N = 0$.

(2) $\Rightarrow$ (1) Since $\mathrm{E}(R/N)$ is faithful, $R \subseteq R_N$ and $\mathrm{r}_R(c) = 0$ for any $c \in \mathrm{C}(N)$. Furthermore, R_N is a right flat epimorphic extension of R because $\mathrm{rad}_{E(R/N)}$ is perfect, and R_N is left Artinian by Proposition 2.2, since every submodule of

E(R/N) is finitely annihilated. Then $(R/N)_N = R_N/N_N$ is an Artinian R_N-module, which implies that R/N has finite left Goldie dimension, and so R/N is a semiprime left Goldie ring since every left ideal of R/N is finitely annihilated. Since $N \cdot (R/N)_N = 0$ we must have $N \cdot (R_N/N_N) = 0$ and so $N_N \cdot (R_N/N_N) = 0$, which implies that N_N is an ideal of R_N. By [7, Theorem 3], R satisfies the left Ore condition with respect to C(N), and it follows from [17, Chapter II, Proposition 1.5] that C$(N) \subseteq$ C(0). Since R_N is left Artinian it must be the left classical ring of quotients of R. □

We refer the reader to the Memoir [12] of Gordon and Robson for the definition of Krull dimension, although we list below the several facts which we will utilize. A module with Krull dimension has finite Goldie dimension. If W is an indecomposable injective module over a ring with Krull dimension on the left, then there is a submodule $U \subseteq W$ such that Ann(U) is a prime ideal. If U is a uniform module over a ring with Krull dimension on the left, then $U \supseteq U'$ where U' is a prime submodule. A ring with Krull dimension on the left has the ascending chain condition on prime ideals, and there are only finitely many prime ideals minimal over any ideal.

A nonzero module M is said to be monoform if every nonzero homomorphism from a submodule of M into M is a monomorphism. Any nonzero module over a ring with Krull dimension on the left contains a monoform submodule. The injective envelope of a monoform module is indecomposable. A torsion radical π is said to be prime if it is cogenerated by the injective envelope of a monoform module. If π is a prime torsion radical, then the quotient category it determines is cogenerated by the injective envelope of a simple object and thus has only one isomorphism class of simple objects. For a ring with Krull dimension on the left, the prime torsion radicals of R–Mod are just those defined by the indecomposable injective left R-modules. A prime torsion radical π is said to be an associated prime torsion radical of the module M if $\pi = \mathrm{rad}_{E(U)}$, where U is a monoform submodule of M.

THEOREM 2.5 *Let R be a ring with Krull dimension on the left, and let π be a prime torsion radical of R–Mod defined by an indecomposable injective module ${}_RW$. The following conditions are equivalent.*

(1) *π is a maximal torsion radical of R–Mod.*

(2) *Every submodule of W is finitely annihilated.*

(3) *Every π-torsionfree left R-module is finitely annihilated.*

Proof. (1) $\Rightarrow$ (2) Let X be a submodule of W, with $\mathrm{Ann}(X) = A$. If U is a uniform submodule of R/A, then U is π-torsionfree since R/A is π-torsionfree, and so $\mathrm{E}(U)$ cogenerates a prime torsion radical τ with $\tau \geq \pi$. By assumption π is maximal, so $\tau = \pi$ and π is thus the only associated prime torsion radical of R/A. Since R/A has finite Goldie dimension, it contains an essential direct sum $\oplus_{i=1}^n U_i$ of uniform submodules. Each U_i defines π, so we can assume that every nonzero submodule of each U_i is π-dense. Thus if $B/A \subseteq R/A$ is π-closed, then for a given i either $(B/A) \cap U_i = 0$ or $B/A \supseteq U_i$. Each annihilator $\mathrm{Ann}(x)$ for $x \in X$ is π-closed and $A = \mathrm{Ann}(X) = \cap_{x \in X} \mathrm{Ann}(x)$, so choosing x_i such that $(\mathrm{Ann}(x_i)/A) \cap U_i = 0$ gives a finite subset $\{x_i\}_{i=1}^n$ such that $\cap_{i=1}^n (\mathrm{Ann}(x_i)/A) = 0$, and thus X is finitely annihilated.

(2) $\Rightarrow$ (3) This follows from Lemma 1.3 (b).

(3) $\Rightarrow$ (1) Since R has Krull dimension on the left, there exists a prime ideal P which is the annihilator of a submodule X of W. By assumption, X is finitely annihilated, and so R/P can be embedded in a direct sum W^n of copies of W. This implies that $\mathrm{E}(R/P)$ is isomorphic to a direct summand of W^n, and since W is indecomposable, the Krull-Remak-Schmidt-Azumaya Theorem shows that W is isomorphic to a direct summand of $\mathrm{E}(R/P)$, which implies that π is cogenerated by $\mathrm{E}(R/P)$.

Let Q be any prime ideal of R which is contained in P. Then R/Q is a strongly prime module, so it must be π-torsionfree since it is not π-torsion. (The factor module R/P is π-torsionfree.) Thus $Q = \mathrm{Ann}(X)$ for a submodule $X \subseteq \mathrm{E}(R/P)$. Since X is finitely annihilated by assumption, there is an embedding of R/Q into $\mathrm{E}(R/P)^n$ for some n. It is easy to show that P annihilates a submodule of R/Q, which forces $P \subseteq Q$, and thus P is a minimal prime ideal. By [4] any minimal prime ideal of a ring with Krull dimension defines a maximal torsion radical. $\square$

COROLLARY 2.6 *Let R be a ring with Krull dimension on the left, and let N be the prime radical of R. Then every submodule of* $\mathrm{E}(R/N)$ *is finitely annihilated.*

Proof. Since $N = \cap_{i=1}^{n} P_i$ for minimal prime ideals $P_1, \ldots, P_n$, it follows that $\mathrm{E}(R/N)$ can be embedded in $\oplus_{i=1}^{n}\mathrm{E}(R/P_i)$. Since $\mathrm{E}(R/P_i)$ cogenerates a maximal torsion radical, the result follows from Theorem 2.5 and Lemma 1.3 (d). □

The next result generalizes a well-known theorem for commutative Noetherian rings. The assumption that all submodules of $\mathrm{E}(R/N)$ are finitely annihilated holds for any order in a left Artinian ring, by Theorem 2.4, and for any ring with Krull dimension, by the preceding corollary. Note that in the latter case, the result also follows from [12, Corollary 7.5].

THEOREM 2.7 *Let R be a ring with prime radical N such that every submodule of* $\mathrm{E}(R/N)$ *is finitely annihilated. The following conditions are equivalent.*

(1) *R is left Artinian.*

(2) *For every prime ideal P of R, R/P is left Artinian.*

(3) *Every finitely generated left R-module is finitely annihilated, and every prime ideal of R is maximal.*

Proof. It is obvious that (1) $\Rightarrow$ (3), and if condition (3) holds, then for any prime ideal P, $P = \mathrm{Ann}(S)$ for a simple R-module ${}_RS$. By assumption, S is finitely annihilated, so there exists an embedding $R \to S^n$ for some n, which implies that

R/P is a simple Artinian ring. To show that (2) $\Rightarrow$ (1), we first note that by assumption every left ideal of R/N is finitely annihilated, so R/N is a strongly semiprime ring. As in the proof of Corollary 1.8, this implies that $\mathrm{E}(R/N) = \oplus_{i=1}^{n}\mathrm{E}(R/P_i)$ for the minimal prime ideals $\{P_i\}_{i=1}^{n}$ of R. If P_i is maximal and R/P_i is left Artinian for each i, then $\mathrm{E}(R/N)$ contains a copy of each simple left R-module, so it is a cogenerator for R–Mod. It follows from Proposition 1.4 that R is left Artinian. $\Box$

Let M be an R-module. A prime ideal P of R is said to be an associated prime ideal of M if $P = \mathrm{Ann}(M')$ for a prime submodule $M' \subseteq M$. The set of all associated prime ideals of M will be denoted by $\mathrm{Ass}(M)$.

For a module ${}_RM$, the element $a \in R$ will be called M-regular if $aRm \neq 0$ for all $0 \neq m \in M$. Observe that $a \in R$ is M-regular if and only if it is $\mathrm{E}(M)$-regular. This definition reduces to the usual one if R is commutative, and the next proposition is well-known for commutative Noetherian rings.

PROPOSITION 2.8 *Let R be a ring with Krull dimension on the left, and let M be a left R-module. The element $a \in R$ is M-regular if and only if a is not contained in any associated prime ideal of M.*

Proof. If a is M-regular and $a \in P$, where $P = \mathrm{Ann}(M')$ for a prime submodule $M' \subseteq M$, then $aM' = 0$, a contradiction.

Conversely, if a is not M-regular, then $aRm = 0$ for an element $0 \neq m \in M$. Since R has Krull dimension, Rm contains a cyclic prime submodule Rbm, for some $b \in R$, and $aRbm = 0$ implies that $a \in P = \mathrm{Ann}(Rbm)$, where $P \in \mathrm{Ass}(M)$. □

PROPOSITION 2.9 *Let R be a ring with Krull dimension on the left, let ${}_RM$ be a module with only finitely many associated prime ideals, let σ be the torsion radical cogenerated by* $\mathrm{E}(M)$, *and let I be an ideal of R. Then the following conditions are equivalent.*

(1) *I is σ-dense in R.*

(2) $\mathrm{Hom}_R(R/I, M) = 0$.

(3) *I is not contained in any associated prime ideal of M.*

(4) *I contains an M-regular element.*

Proof. (1) ⇒ (2) If I is σ-dense, then in fact $\mathrm{Hom}_R(R/I, \mathrm{E}(M)) = 0$.

(2) ⇒ (3) If $I \subseteq P$ and $P = \mathrm{Ann}(X)$ for $0 \neq X \subseteq M$, then it is possible to define a nonzero homomorphism $R/I \to R/P \to X \to M$.

(3) ⇒ (4) Since I is not contained in any associated prime ideal of M, it is not contained in their union. Thus I contains an element which is not in any associated prime ideal, and the desired conclusion follows from Proposition 2.8.

(4) ⇒ (1) If $0 \neq f : A/I \to M$ for a left ideal $A \supset I$, then $I \cdot f(A/I) = 0$, which is impossible if I contains an M-regular

element. This implies that $\mathrm{Hom}_R(R/I, \mathrm{E}(M)) = 0$, and so I is σ-dense. □

PROPOSITION 2.10 *Let R be a ring with Krull dimension on the left, let ${}_RM$ be a module with finite Goldie dimension, and let σ be the torsion radical cogenerated by* $\mathrm{E}(M)$. *If ${}_RN$ is finitely annihilated, then N is σ-torsion if and only if* $\mathrm{Ann}(N)$ *contains an M-regular element.*

Proof. If M has finite Goldie dimension, then $\mathrm{E}(M) = \mathrm{E}(U_1) \oplus \cdots \oplus \mathrm{E}(U_n)$, where each U_i is a uniform R-module. Since R has Krull dimension, $\mathrm{Ass}(U_i) \neq \emptyset$ and each $\mathrm{Ass}(U_i)$ consists of a unique prime ideal since $L \cap K \neq 0$ for any nonzero submodules L and K of U. Because $\mathrm{Ass}(M) = \mathrm{Ass}(\mathrm{E}(M))$, $\mathrm{Ass}(M)$ is finite.

If N is finitely annihilated and N is σ-torsion, then $\mathrm{Ann}(N)$ must be σ-dense. Thus $\mathrm{Ann}(N)$ contains an M-regular element by Proposition 2.9.

Conversely, suppose that $0 \neq f : N' \to M$ for some submodule $N' \subseteq N$. Then $N'/\ker(f)$ has an associated prime ideal P, which must also be an associated prime ideal of M. This is a contradiction since $\mathrm{Ann}(N) = P$. □

PROPOSITION 2.11 *Let R be a ring with Krull dimension on the left, let ${}_RM$ be a module with finite Goldie dimension, and let σ be the torsion radical cogenerated by* $\mathrm{E}(M)$. *If every finitely generated submodule of ${}_RN$ is finitely annihilated,* *then N is σ-torsionfree if and only if every M-regular element is N-regular.*

Proof. The module N is σ-torsionfree if and only if no cyclic submodule is σ-torsion. By Proposition 2.10 this happens if and only if $aRx \neq 0$ for all $0 \neq x \in N$ and all M-regular elements $a \in R$. $\square$

THEOREM 2.12 *Let R be a ring with Krull dimension on the left, and let N be the prime radical of R. The following conditions are equivalent.*

(1) *Every associated prime torsion radical of R is maximal.*

(2) *Every submodule of* $\mathrm{E}(R)$ *is finitely annihilated.*

(3) *Every finitely generated left ideal of R is finitely annihilated and every R/N-regular element of R is R-regular.*

(4) $\mathrm{E}(R/N)$ *is faithful.*

Proof. Since R has finite left Goldie dimension, $\mathrm{E}(R) = \oplus_{i=1}^{n}\mathrm{E}(U_i)$, where U_i is a uniform left ideal of R and defines an associated prime torsion radical of R.

(1) $\Rightarrow$ (4) Let $N = \cap_{i=1}^{n} P_i$ be the intersection of the minimal prime ideals of R. By [13, Proposition 9], R/N is an essential submodule of $R/P_1 \oplus \ldots \oplus R/P_m$. Hence $\mathrm{E}(R/N) = \mathrm{E}(R/P_1) \oplus \ldots \oplus \mathrm{E}(R/P_m)$, and since $R/P_j \subseteq \mathrm{E}(R/N)$, $\mathrm{rad}_{E(R/N)} \leq \mathrm{rad}_{E(R/P_j)}$ for any minimal prime ideal P_j. By [4, Theorem 4.6] $\mathrm{rad}_{E(U_i)} = \mathrm{rad}_{E(R/P_j)}$ for some minimal prime ideal, and so

$\mathrm{rad}_{E(R/N)}(\mathrm{E}(U_i)) \subseteq \mathrm{rad}_{E(U_i)}(\mathrm{E}(U_i)) = 0$ and hence $\mathrm{Ann}(\mathrm{E}(R/N)) = \mathrm{rad}_{E(R/N)}(R) = 0$.

(4) $\Rightarrow$ (2) Let X be a submodule of $\mathrm{E}(R)$. Then $\mathrm{rad}_{E(R)} \geq \mathrm{rad}_{E(R/N)}$ since $\mathrm{E}(R/N)$ is faithful, so $\mathrm{rad}_{E(R/N)}(X) \subseteq \mathrm{rad}_{E(R)}(X) = 0$ and X is contained in a direct product of copies of $\mathrm{E}(R/N)$. By Corollary 2.6 and Lemma 1.3 (b), X is finitely annihilated.

(2) $\Rightarrow$ (1) Each associated prime torsion radical of R is cogenerated by $\mathrm{E}(U_i)$, for some i, so it is maximal by Theorem 2.5 since every submodule of $\mathrm{E}(U_i)$ is finitely annihilated.

(4) $\Rightarrow$ (3) By (4) $\Rightarrow$ (2) above, every submodule of R is finitely annihilated and R/N has finite left Goldie dimension. By hypothesis $\mathrm{rad}_{E(R/N)}(R) = 0$ and so condition (3) follows from Proposition 2.11.

(3) $\Rightarrow$ (4) Proposition 2.11 implies that $\mathrm{rad}_{E(R/N)}(R) = 0$, and so $\mathrm{E}(R/N)$ is faithful. □

3 Linear topologies and finitely generated, finitely annihilated modules

If τ is a torsion preradical, and M is τ-torsion, then any left annihilator $\mathrm{Ann}(m)$, $m \in M$, must be τ-dense; thus if $D \supseteq \cap_{i=1}^{k} \mathrm{Ann}(m_i)$, $m_i \in M$, then D must be τ-dense. For $a \in R$, $\{r \mid ra \in \cap_{i=1}^{k} \mathrm{Ann}(m_i)\} = \cap_{i=1}^{k} \mathrm{Ann}(am_i)$, so the set of all left ideals D such that $D \supseteq \cap_{i=1}^{k} \mathrm{Ann}(m_i)$ for some finite set $m_1, \ldots, m_k \in M$ defines a linear topology, and the corresponding torsion preradical $\mathrm{Rad}^{\mathrm{M}}$ is the smallest torsion preradical for which M is torsion. It can be shown that for an injective module ${}_RW$, $\mathrm{Rad}^{\mathrm{M}}(W)$ is just the sum in W of all homomorphic images of M, and then for any module ${}_RX$, $\mathrm{Rad}^{\mathrm{M}}(X) = X \cap \mathrm{Rad}^{\mathrm{M}}(\mathrm{E}(X))$. Note that if τ is any torsion preradical, then $\tau = \mathrm{Rad}^{\mathrm{M}}$ for the direct sum $M = \oplus_{D \in \mathcal{D}} R/D$ taken over the set $\mathcal{D}$ of τ-dense left ideals of R. Since the quasi-injective hull $\overline{M}$ of M is the sum of homomorphic images of M in $\mathrm{E}(M)$, for any torsion preradical τ we must have $\tau(M) = M$ if and only if $\tau(\overline{M}) = \overline{M}$. Thus $\mathrm{Rad}^{\mathrm{M}} = \mathrm{Rad}^{\overline{M}}$, and the following proposition gives another proof that M is finitely annihilated if and only if $\overline{M}$ is finitely annihilated.

PROPOSITION 3.1 *The following conditions are equivalent for the module* ${}_RM$.

(1) *M is finitely annihilated.*

(2) *The intersection of all* $\mathrm{Rad}^{\mathrm{M}}$*-dense left ideals of R is* $\mathrm{Rad}^{\mathrm{M}}$*-dense.*

(3) *The class of* $\mathrm{Rad}^{\mathrm{M}}$*-torsion modules is closed under direct products.*

Proof. (1) $\Leftrightarrow$ (2) The intersection of $\mathrm{Rad}^{\mathrm{M}}$-dense left ideals is $\mathrm{Ann}(M)$, so condition (2) holds if and only if $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \ldots, m_k)$ for some $m_1, \ldots, m_k \in M$.

(2) $\Rightarrow$ (3) If $\mathrm{Ann}(M)$ is $\mathrm{Rad}^{\mathrm{M}}$-dense, then ${}_RX$ is $\mathrm{Rad}^{\mathrm{M}}$-torsion if and only if $\mathrm{Ann}(M)$ annihilates X.

(3) $\Rightarrow$ (2) If $\mathcal{D}$ is the set of $\mathrm{Rad}^{\mathrm{M}}$-dense left ideals of R, then $R/\cap_{D\in\mathcal{D}} D$ can be embedded in $\prod_{D\in\mathcal{D}} R/D$, so if the latter is $\mathrm{Rad}^{\mathrm{M}}$-torsion, then $\cap_{D\in\mathcal{D}} D \in \mathcal{D}$. □

COROLLARY 3.2 *The following properties of a module ${}_RM$ are preserved by a category equivalence $\mathcal{F} : R\text{–}Mod \to S\text{–}Mod$.*

(*a*) *M is finitely annihilated.*

(*b*) *Every submodule of M is finitely annihilated.*

Proof. (a) The class of $\mathrm{Rad}^{\mathrm{M}}$-torsion modules is the smallest class of modules which contains M and is closed under homomorphic images, direct sums and submodules. Thus the class of $\mathrm{Rad}^{\mathrm{M}}$-torsion modules corresponds under $\mathcal{F}$ to the class of $\mathrm{Rad}^{\mathcal{F}(M)}$-torsion modules, and if the former class is closed under direct products, then the latter class satisfies the same condition.

(b) This follows immediately from (a). □

The next corollary follows from Proposition 1.4. Note that a condition similar to condition (3) can be used to characterize right perfect rings (see [17, Chapter VIII, Corollary 6.3]).

COROLLARY 3.3 *The following conditions are equivalent.*

(1) *R is left Artinian.*

(2) *For any linear topology $\mathcal{D}$ of R, $\cap_{D\in\mathcal{D}} D \in \mathcal{D}$.*

(3) *Every hereditary pretorsion class of $R\text{–}Mod$ is closed under direct products.*

PROPOSITION 3.4 *Let $_RM$ be a finitely generated module. Then M is finitely annihilated if and only if* $\mathrm{Rad}^{\mathrm{M}}$ *is a bounded torsion preradical.*

Proof. Recall that a torsion preradical τ is said to be bounded if every τ-dense left ideal contains a (two-sided) τ-dense ideal.

If M is finitely annihilated, then $\mathrm{Ann}(M)$ is $\mathrm{Rad}^{\mathrm{M}}$-dense and is contained in every $\mathrm{Rad}^{\mathrm{M}}$-dense left ideal.

Conversely, let $M = \sum_{i=1}^{n} Rm_i$. Then $\cap_{i=1}^{n}\mathrm{Ann}(m_i)$ is $\mathrm{Rad}^{\mathrm{M}}$-dense, so by assumption it must contain an ideal I which is $\mathrm{Rad}^{\mathrm{M}}$-dense. Then $\mathrm{Ann}(M) \supseteq I \supseteq \cap_{i=1}^{n} \mathrm{Ann}(x_i)$ for some $x_1, \ldots, x_k \in M$, so $\mathrm{Ann}(M) = \cap_{i=1}^{n} \mathrm{Ann}(x_i)$ and M is finitely annihilated. □

The conditions of the following proposition are satisfied for several important classes of rings: (a) R is left Noetherian and fully left bounded (see Gabriel [10] and Cauchon [8]); (b) R is finitely generated as a module over its center; and (c) R is a left duo ring, that is, every left ideal of R is two-sided.

PROPOSITION 3.5 *Every finitely generated left R-module is finitely annihilated if and only if every linear topology of R–Mod is bounded.*

Proof. Assume that every finitely generated left R-module is finitely annihilated, and let $\{D_i\}_{i=1}^{n}$ be a set of τ-dense left ideals for a torsion preradical τ of R–Mod. Then by assumption

$\oplus_{i=1}^{n} R/D_i$ is finitely annihilated, so $\cap_{i=1}^{n} D_i \supseteq \mathrm{Ann}(\oplus_{i=1}^{n} R/D_i)$, and the latter is τ-dense since it must be equal to a finite intersection of left annihilators of elements of $\oplus_{i=1}^{n} R/D_i$.

The converse follows from the previous proposition. □

LEMMA 3.6 *Let R be a (left) strongly prime ring. The module ${}_RM$ is faithful and finitely annihilated if and only if* $\mathrm{Z}(M) \neq M$.

Proof. If M is faithful and finitely annihilated, then there exists an embedding $R \to M^n$ for some n, and $\mathrm{Z}(M) = M$ implies $\mathrm{Z}(R) = R$, a contradiction.

Conversely, if $\mathrm{Z}(M) \neq M$, then there exists a homomorphism $0 \neq f : M \to \mathrm{E}(R)$, since by assumption R is strongly prime and hence $\mathrm{Z}(R) = 0$. It is evident that M must then be faithful and finitely annihilated, since every submodule of $\mathrm{E}(R)$ is faithful and finitely annihilated. □

PROPOSITION 3.7 *Let R be a prime ring. Then every finitely generated, faithful left R-module is finitely annihilated if and only if R is (left) strongly prime and every essential left ideal of R contains a nonzero (two-sided) ideal.*

Proof. Assume that every finitely generated, faithful left R-module is finitely annihilated. It follows from Proposition 1.7 that R is strongly prime, since every nonzero left ideal is faithful. Let A be an essential left ideal of R. If R/A is faithful, then it must be finitely annihilated, and the existence of an embedding $R \to (R/A)^k$ gives a contradiction, since $\mathrm{Z}(R) = 0$ but $\mathrm{Z}(R/A) = R/A$. Thus $0 \neq \mathrm{Ann}(R/A) \subseteq A$.

Conversely, suppose that M is finitely generated and faithful, with $M = \sum_{i=1}^{n} Rm_i$. If M is not finitely annihilated, then by Lemma 3.6, $\mathrm{Z}(M) = M$ and so $\cap_{i=1}^{n} \mathrm{Ann}(m_i)$ is essential. Thus $\cap_{i=1}^{n} \mathrm{Ann}(m_i) \supseteq A \neq 0$ for some ideal A, and $AM = 0$ contradicts the assumption that M is faithful. □

Recall that a left Noetherian ring is called fully left bounded if all of its prime factor rings satisfy the equivalent conditions of Proposition 3.7. Cauchon [8] has shown that a left Noetherian ring is fully left bounded if and only if every finitely generated left module is finitely annihilated. Schelter [15] has shown recently that if R is left and right Noetherian and fully bounded, then any finitely generated, essentially Artinian left R-module is Artinian. Using the same proof, we can restate his theorem in the following manner.

PROPOSITION 3.8 *Let R be a right Noetherian ring, and let ${}_RM$ be a Noetherian module which contains an essential Artinian submodule. The following conditions are equivalent.*

(1) *Every submodule of a factor module of M is finitely annihilated.*

(2) *$R/\mathrm{Ann}(M)$ is fully left bounded.*

(3) *M is Artinian.*

To conclude this section, we note that for a left primitive ring R, every finitely generated left R-module is finitely annihilated if and only if R is simple Artinian.

4 Rings whose left ideals are finitely annihilated

PROPOSITION 4.1 *The following conditions are equivalent for the ring R.*

(1) *Every left ideal of R is finitely annihilated.*

(2) *Every ideal of R is finitely annihilated as a left ideal.*

(3) *Every torsionless left R-module is finitely annihilated.*

Proof. (2) $\Rightarrow$ (1) This follows from Lemma 1.3 (c).

(1) $\Rightarrow$ (3) Since a module is torsionless if and only if it can be embedded in a direct product of copies of R, this follows from Lemma 1.3 (b).

(2) $\Rightarrow$ (1) An ideal is certainly torsionless. □

COROLLARY 4.2 *The condition that every left ideal of a ring is finitely annihilated is a Morita invariant condition.*

Proof. The classes of torsionless and finitely annihilated modules are both invariant under an equivalence of module categories. □

PROPOSITION 4.3 *Let A be the left annihilator of a left ideal of R. If every left ideal of R is finitely annihilated, then the same property holds for the ring R/A.*

Proof. Let I be a left ideal of R such that $A = \ell(I)$. Then there exists an embedding $R/A \to I^n$ for some n, and the result follows from Lemma 1.3 (d). □

On the one hand, the condition that every left ideal is finitely annihilated can be viewed as a weak form of the descending chain condition for left annihilators. In this regard, it is interesting to note that if every left ideal of R is finitely annihilated and R is a cogenerator in R–Mod, then R is left Artinian by Proposition 1.4. On the other hand, we might expect some similarities with fully left bounded, left Noetherian rings. In this regard, we have the following proposition. Note that the assumptions are satisfied not only by any fully left bounded, left Noetherian ring, but also by any left and right Noetherian ring, and by any order in an Artinian ring. Recall that for a fully left bounded, left Noetherian ring there is a one-to-one correspondence between prime ideals and indecomposable injective modules (which correspond to prime torsion radicals).

PROPOSITION 4.4 *Let R be a ring with finite left Goldie dimension such that every ideal of R is finitely annihilated both as a left ideal and as a right ideal. Then every left ideal of R contains a monoform left ideal, and there is a one-to-one correspondence* *between associated prime ideals of ${}_RR$ and associated prime torsion radicals of ${}_RR$.*

Proof. By Proposition 1.2, the descending chain condition must hold for right annihilators of right ideals of R, since every right ideal of R is finitely annihilated. It follows that the ascending chain condition must hold for annihilator of left ideals of R, so any left ideal $A \subseteq R$ contains a left ideal B such that $P = \mathrm{Ann}(B)$ is a maximal left annihilator ideal. Then by a standard argument P is a prime ideal, and there exists an embedding $R/P \to B^n$ for some n, since B is finitely annihilated. Thus R/P has finite left Goldie dimension and every left ideal of R/P is finitely annihilated (by Lemma 1.3 (d)), so R/P is a prime Goldie ring, and since uniform left ideals of a prime Goldie ring are monoform it can be checked that B must contain a monoform left ideal. If U is a monoform left ideal of R, then the above argument shows that U has an associated prime ideal P, and moreover that $\mathrm{rad}_{E(U)} = \mathrm{rad}_{E(R/P)}$. □

In [6] it was shown that if R is a commutative ring in which every faithful ideal is finitely annihilated, then the polynomial ring $R[x]$ enjoys the same property. If we delete "faithful" from the above statement, we raise the following question.

QUESTION (1). If every left ideal of R is finitely annihilated, does the same condition hold for $R[x]$?

This question appears to be open even for commutative rings. An affirmative answer to the following question, which seems to have some independent interest, would give an affirmative answer to the first question (for commutative rings).

QUESTION (2). If every ideal of the commutative ring R is finitely annihilated, can R be embedded in an Artinian ring?

Another interesting line of questioning arises in the matter of whether nil subrings of a ring with certain chain conditions are nilpotent (we temporarily relax the requirement that rings have an identity element). A well-known theorem of Herstein and Small states that nil subrings are nilpotent in rings which satisfy both chain conditions on left annihilators. The following example due to Small, which appeared in [11], shows that we cannot weaken the chain condition on annihilators to the condition that ideals are left and right finitely annihilated.

Let F be a field of characteristic p, and let R be an finitely generated nil F-algebra which is not locally nilpotent. Let $\overline{R} = R/I$, where I is maximal with respect to the property that $R^n \not\subseteq I$ for all positive integers n. Let $S = \mathbb{Z}_p \times \overline{R}$ be the usual characteristic p adjunction of unity. Then it is easy to see that S is a prime ring and that every ideal of S is finitely annihilated on both sides. Finally, $\overline{R}$ is a nil but not nilpotent ideal of S.

A theorem of Lanski states that nil subrings of left Goldie rings are nilpotent. If we replace the ascending chain condition on left annihilators with the condition that every ideal is finitely annihilated on the left, we have the following question.

QUESTION (3). If R is a ring with finite left Goldie dimension such that every ideal is finitely annihilated on the left (right), then are nil subrings (ideals) of R necessarily nilpotent?

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