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MAXIMAL TORSION RADICALS OVER RINGS WITH FINITE REDUCED RANK

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For a left Noetherian ring R with prime radical N, the reduced rank of R has been defined by Goldie [9] as follows. Let Q be the semisimple classical ring of left quotients of R/N, and assume that k is the index of nilpotence of N. Then the reduced rank of R, denoted by $\rho(R)$, is defined by the formula

$$\rho(R) = \sum_{i=1}^{k} \ell(Q \otimes_R (N^{i-1}/N^i)) ,$$

where $\ell(X)$ denotes the length of a Q-module X.

Let γ denote the torsion radical of R–Mod cogenerated by $\mathrm{E}(R/N)$, with the associated quotient functor denoted by $Q_\gamma : R\text{–}Mod \to R\text{–}Mod/\gamma$. If R is left Noetherian, then it follows from a result of Jategaonkar [11, Proposition 1.9] that $Q_\gamma(R)$ has finite length as an object in the quotient category R–Mod/γ. This length is precisely the reduced rank of R [7, Proposition 2]. Thus it is possible to give the following more general definition. The ring R is said to have *finite reduced rank* (on the left) if $Q_\gamma(R)$ has finite length in the quotient category R–Mod/γ cogenerated by $\mathrm{E}(R/N)$.

Lenagan [12] has shown that Goldie's definition of reduced rank can be extended to any ring with Krull dimension. If also follows from [6, Proposition 6] that if R has Krull dimension, then it has finite reduced rank in the general sense. Furthermore, [7, Theorem 4] shows that the ring R is a left order in a left Artinian ring if and only if R has finite reduced rank (on the left) and satisfies the regularity condition. Thus the class of rings with finite reduced rank is quite extensive. In Theorem 4 below it will be shown that if R has finite reduced rank and S is Morita equivalent to R, then S also has finite reduced rank.

The Walkers [15, Theorem 1.29] showed that for a commutative ring R there is a one-to-one correspondence between maximal torsion radicals of R–Mod and minimal prime ideals of R. This correspondence holds for any ring with Krull dimension by [4, Theorem 4.6], and furthermore, in this case every torsion radical is contained in a maximal torsion radical. Theorem 2 below shows that these conditions hold for any ring with finite reduced rank whose prime radical is right T-nilpotent. In fact, rings with finite reduced rank can be characterized by conditions involving maximal torsion radicals (see Theorem 1).

Throughout the paper, R will be assumed to be an associative ring with identity element, and all modules will be assumed to be unital R-modules. The category of left R-modules will be denoted by $R\text{–}Mod$, the direct sum of n isomorphic copies of a module ${}_RM$ will be denoted by M^n, and the injective envelope of M will be denoted by $\mathrm{E}(M)$. The reader is referred to the book by Stenström [14] for definitions and results on quotient categories and torsion radicals.

If σ is a torsion radical (called a left exact radical by Stenström), then the quotient category it determines will be denoted by $R\text{–}Mod/\sigma$, with the exact quotient functor denoted by $Q_\sigma : R\text{–}Mod \to R\text{–}Mod/\sigma$. A module ${}_RM$ is called σ-torsionfree if $\sigma(M) = (0)$, and σ-torsion if $\sigma(M) = M$; a submodule $M' \subseteq M$ is called σ-closed if M/M' is σ-torsionfree, and σ-dense if M/M' is σ-torsion. Recall that the subobjects of $Q_\sigma(M)$ in $R\text{–}Mod/\sigma$ correspond to the σ-closed submodules of M [14, Chapter IX, Corollary 4.4]. Thus in the quotient category $R\text{–}Mod/\sigma$, the object $Q_\sigma(M)$ has finite length if and only if M satisfies both ascending and descending chain conditions on the set of σ-closed submodules. In this case, $M/\sigma(M)$ must have finite uniform dimension.

If ${}_RX$ is an injective module, then X defines a torsion radical rad_X by setting

$$\mathrm{rad}_X(M) = \{m \in M \mid f(m) = 0 \ \text{ for all } \ f \in \mathrm{Hom}_R(M, X)\}$$

for any module M in $R\text{–}Mod$. If $\sigma = \mathrm{rad}_X$, then $X = Q_\sigma(X)$ is a cogenerator for $R\text{–}Mod/\sigma$, and σ is said to be the torsion radical cogenerated by X.

The torsion radical σ is said to be *prime* if $\sigma = \mathrm{rad}_{E(M)}$ for a module ${}_RM$ such that $Q_\sigma(M)$ is a simple object in $R\text{–}Mod/\sigma$. Equivalently, $R\text{–}Mod/\sigma$ has a cogenerator which is the injective envelope of a simple object. In this case, any cogenerator of $R\text{–}Mod/\sigma$ contains an isomorphic copy of the simple object. Note that if P is a prime Goldie ideal of R (that is, R/P is a prime left Goldie ring), then $\mathrm{rad}_{E(R/P)}$ is a prime torsion radical.

If σ and τ are torsion radicals, then $\sigma \leq \tau$ if $\sigma(M) \subseteq \tau(M)$ for each module M in $R\text{–}Mod$. Note that $\sigma = \mathrm{rad}_{E(X)}$ is the largest torsion radical such that X is σ-torsion. If σ is a proper torsion radical (that is, σ is not the identity functor on $R\text{–}Mod$), then σ is said to be *maximal* if $\sigma \leq \tau$ implies $\sigma = \tau$ for all proper torsion radicals τ. By [3, Theorem 2.4], σ is maximal if and only if the quotient category $R\text{–}Mod/\sigma$ is nonzero and each nonzero σ-torsionfree injective module is a cogenerator of $R\text{–}Mod/\sigma$.

THEOREM 1 *The following conditions are equivalent.*

(1) The ring R has finite reduced rank (on the left).

(2) There exist maximal torsion radicals $\mu_1, \ldots, \mu_n$ such that the localization of R has finite length in $R\text{–Mod}/\mu_i$, for $i = 1, \ldots, n$, and such that for each prime ideal P of R, $\mathrm{rad}_{E(R/P)} \leq \mu_i$ for some i.

Proof. (1) ⇒ (2). Let N be the prime radical of R, and let $\gamma = \mathrm{rad}_{E(R/N)}$. By assumption $Q_\gamma(R)$ has finite length in R–Mod/γ, so R/N is a semiprime left Goldie ring by [6, Proposition 2]. Let P be a minimal prime ideal of R, and let $\mu = \mathrm{rad}_{E(R/P)}$. Then $\gamma \le \mu$, so every μ-closed left ideal of R is γ-closed, which implies that R/P is a left Goldie ring, and so μ is a prime torsion radical. Furthermore, $Q_\mu(R)$ must have finite length in R–Mod/μ, and so each μ-torsionfree injective module contains a submodule which maps to a simple object in the quotient category. This shows that any μ-torsionfree injective module cogenerates R–Mod/μ, and therefore μ is a maximal torsion radical.

Let $P_1, \ldots, P_n$ be the minimal prime ideals of R, and let μ_i be the torsion radical cogenerated by $\mathrm{E}(R/P_i)$. If P is any prime ideal of R, then $P \supseteq P_i$ for some i, and so $\mathrm{rad}_{E(R/P)} \le \mu_i$ by [3, Lemma 2.5] since P_i is a prime Goldie ideal.

(2) ⇒ (1). Assume that condition (2) holds. If R has finite length with respect to μ_i, then by [6, Theorem 7], $\mu_i = \mathrm{rad}_X$ for an injective module $X \cong \oplus_{\alpha \in I} \mathrm{E}(R/P_\alpha)$, where each ideal P_α is a prime Goldie ideal. If P_i is any one of the prime ideals in the decomposition of X, then $\mathrm{E}(R/P_i)$ cogenerates a torsion radical larger than or equal to μ_i, so the maximality of μ_i implies that $P_\alpha = P_i$ for all $\alpha \in I$.

If Q is a minimal prime ideal of R, then $\mathrm{rad}_{E(R/Q)} \le \mu_i$ for some i, which implies that $P_i \subseteq Q$ by [3, Lemma 3.5], and so $P_i = Q$. Thus each minimal prime ideal is represented in the set $P_1, \ldots, P_n$, and each minimal prime ideal is a prime Goldie ideal since R has finite length with respect to μ_i. If P_i were not a minimal prime for some i, then $P_i \supseteq P_j$ for some minimal prime P_j with $j \ne i$. The maximality of μ_i would then imply that $P_i = P_j$, a contradiction. Thus $P_1, \ldots, P_n$ is the set of minimal prime ideals of R. Since the prime torsion radicals $\mu_i, \ldots, \mu_n$ are incomparable and R has finite length with respect to μ_i, for each i, it follows from [10, Theorem 3.6] that R has finite length with respect to $\mathrm{rad}_{E(R/N)} = \cap_{i=1}^n \mu_i$, and thus R has finite reduced rank. □

An ideal I of R is said to be right *T-nilpotent* in case for each sequence $a_1, a_2, \ldots$ of elements in I there exists an integer n such that $a_n \cdots a_2 a_1 = 0$. The following example shows that the prime radical of a ring with finite reduced rank may be T-nilpotent but not nilpotent. (Example 2 of [7] shows that the prime radical need not even be T-nilpotent.)

Example. Let $R = F[x_1, x_2, \ldots]$ be the ring of polynomials in infinitely many indeterminates over a field F, and let I be the ideal generated by $x_2^2, x_3^3, \ldots$ and the products $x_i x_j$, for $i \ne j$. The ideal N generated by $x_2, x_3, \ldots$ is a prime ideal, and since each element of N is nilpotent in R/I, it follows that N/I is the prime radical of R/I. Furthermore, $x_1 \notin N$ but $x_1 x_j \in I$ for $j \ne 1$, and so N/I is torsion with respect to the torsion radical γ of R/I–Mod cogenerated by $\mathrm{E}_{R/I}(R/N)$. Thus the localization $Q_\gamma(R/I)$ is just the quotient field of R/N, and $\rho(R/I) = 1$. It can be immediately checked that N/I is T-nilpotent. Thus R/I is a ring with finite reduced rank whose prime radical is T-nilpotent but not nilpotent.

THEOREM 2 *The following conditions are equivalent for a ring R with prime radical N.*
(1) The ring R has finite reduced rank (on the left) and N is right T-nilpotent.
(2) (i) The category R–Mod has maximal torsion radicals $\mu_1, \ldots, \mu_n$ such that for $i = 1, \ldots, n$ the ring R has finite length with respect to μ_i;
(ii) for each torsion radical σ of R–Mod, $\sigma \leq \mu_i$ for some i, $1 \leq i \leq n$;
(iii) the maximal torsion radicals of R–Mod correspond to minimal prime ideals of R.

Proof. (1) $\Rightarrow$ (2). Condition (i) follows from Theorem 1, which also shows that R/N is a left Goldie ring. By [5, Theorem 1.8], conditions (ii) and (iii) hold for R/N, and then they hold for R by [5, Proposition 2.7] since N is right T-nilpotent.

(2) $\Rightarrow$ (1). By [5, Proposition 2.7], conditions (ii) and (iii) imply that N is right T-nilpotent. Conditions (i) and (ii) imply that R has finite reduced rank, by Theorem 1. □

Goldman has shown [10, Theorem 5.10] that if R is a left Noetherian ring, then R is left Artinian if and only if every prime torsion radical is maximal. The next theorem extends this result to certain rings with finite reduced rank.

THEOREM 3 *Let R be a ring with finite reduced rank (on the left) such that the prime radical of R is right T-nilpotent. Then R is left Artinian if and only if every prime torsion radical of R–Mod is maximal.*

Proof. If R is left Artinian, then every prime torsion radical is maximal by [10, Theorem 5.10]. Conversely, if every prime torsion radical is maximal, then each simple module defines a maximal torsion radical. By Theorem 2, there are only finitely many maximal torsion radicals, and so R–Mod has only finitely many nonisomorphic simple modules $S_1, \ldots, S_n$. If the torsion radical μ_i is cogenerated by $\mathrm{E}(S_i)$, $i = 1, \ldots, n$, then ${}_RR$ has finite length with respect to $\mu = \cap_{i=1}^n \mu_i$. Since $S_1, \ldots, S_n$ is a complete set of representatives of the equivalence classes of simple modules, $\oplus_{i=1}^n \mathrm{E}(S_i)$ is a cogenerator for R–Mod, which implies that μ is the zero functor. This shows that ${}_RR$ has finite length. □

THEOREM 4 *Let R and S be Morita equivalent rings. If R has finite reduced rank (on the left), then so does S.*

Proof. The proof will make use of the characterization of rings with finite reduced rank given in Theorem 1. Let $F : R$–$Mod \to S$–Mod be an equivalence of categories. If ${}_RX$ is cogenerated by ${}_RY$, then by [1, Proposition

21.6], ${}_SF(X)$ is cogenerated by ${}_SF(Y)$. Since F preserves injective modules, this shows that F preserves the lattice of torsion radicals. Furthermore, corresponding torsion radicals determine isomorphic quotient categories, and so S must have finitely many maximal torsion radicals, and it must have finite length with respect to each of these. The proof can be completed by showing that F preserves torsion radicals of the form $\mathrm{rad}_{E(R/P)}$, where P is a prime ideal.

As shown by [1, Proposition 21.11], ideals of R correspond to ideals of S by assigning to the ideal I of R the ideal $\mathrm{Ann}_S(F(R/I))$ of S. The characterization of prime ideals given in [2, Theorem 2] shows that if P is a prime ideal of R, then the R-module R/P has the property that if ${}_RX$ is cogenerated by R/P, then R/P is cogenerated by X. Furthermore, if ${}_SY$ has the property that ${}_SY$ is cogenerated by ${}_SX$ whenever X is cogenerated by Y, then $\mathrm{Ann}_S(Y)$ is a prime ideal. This shows that under the one-to-one correspondence between ideals of R and S defined above, if P is a prime ideal of R, then there is a prime ideal Q of S such that $F(R/P)$ cogenerates and is cogenerated by S/Q. If follows that $\mathrm{E}(S/Q)$ and $F(\mathrm{E}(R/P)) \cong \mathrm{E}(F(R/P))$ cogenerate the same torsion radical, so that the equivalence between R–*Mod* and S–*Mod* preserves the torsion radicals corresponding to prime ideals. □

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