M-INJECTIVE MODULES AND PRIME M-IDEALS

John A. Beachy

Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115

ABSTRACT: For a left R-module M, I will identify certain submodules of M that play a role analogous to that of prime ideals. Using this definition, I will look at conditions on the module M which imply that there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules and "prime M-ideals".

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(1) Radicals

R will be an associative ring with identity, and ${}_RM$ will be a fixed left R-module.

A *radical* of R–Mod is a subfunctor of the identity with

$\rho(X/\rho(X)) = (0)$ for all ${}_RX$.

The radical of R–Mod *cogenerated* by a class $\mathcal{C}$ is

$\text{rad}_{\mathcal{C}}(X) = \bigcap_{f:X\to C} \ker(f) = \text{Ann}_X(\mathcal{C})$.

If $\mathcal{C} = \{{}_RW\}$, we use the notation

$\text{rad}_W(X) = \bigcap_{f:X\to W} \ker(f) = \text{Ann}_X(W)$

Note that rad_W is the largest radical for which W is torsionfree.

Thrm [1973]. There is a one-to-one correspondence between maximal radicals of R–Mod and prime ideals of R.

The prime ideal corresponding to a maximal radical μ is $\mu(R)$; the maximal radical corresponding to a prime ideal P is $\text{rad}_{R/P}$.

(2) The subcategory $\sigma[M]$

$\sigma[M]$ is the full subcategory of R–Mod that contains all ${}_RX$ isomorphic to a submodule of an M-generated module.

$\sigma[M] = R$–Mod iff R belongs to $\sigma[M]$ iff M is faithful and finitely annihilated.

$\sigma[M]$ is closed under homomorphic images, submodules, and direct sums. It has pullbacks, pushouts, a generator, and products. Injective modules and injective envelopes exist.

Definition 1.2. The submodule $N \subseteq M$ is called an M-ideal if there is a class $\mathcal{C}$ in $\sigma[M]$ with $N = \mathrm{Ann}_M(\mathcal{C})$.

Proposition [Bican, Jambor, Kepka, Němec: 1977] The following are equivalent for a submodule $N \subseteq M$.

(1) N is an M-ideal;
(2) there is a radical ρ of R–Mod with $N = \rho(M)$;
(3) $g(N) = (0)$ for all $g \in \mathrm{Hom}_R(M, (M/N))$;
(4) $N = \mathrm{Ann}_M(M/N)$.

(3) Products $N \cdot X$

If I is any left ideal of R, setting $\rho(X) = IX$ defines a radical. If N is an M-ideal, then $\tau(X) = \sum_{f:M\to X} f(N)$ need not define a radical. But $\tau(X) = (0)$ iff $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, X)$, and this class defines the smallest radical ρ with $\tau \leq \rho$.

Definition 1.5. Let N be a submodule of M. For each module ${}_RX$ we define $N \cdot X = \mathrm{Ann}_X(\mathcal{C})$, where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = (0)$ for all $f \in \mathrm{Hom}_R(M, W)$.

Proposition 1.6. Let N be a submodule of M. Then for any module ${}_RX$ we have $N \cdot X = (0)$ if and only if $N \subseteq \mathrm{Ann}_M(X)$.

Corollary 1.7. If N is a submodule of M, then N is an M-ideal if and only if $N \cdot (M/N) = (0)$.

(4) M-prime modules

${}_RX$ is *prime* if $X \neq 0$ and $\mathrm{Ann}_R(Y) = \mathrm{Ann}_R(X)$, for all nonzero submodules $Y \subseteq X$.

Equivalently, ${}_RX$ is prime iff $IY = (0)$ implies $IX = (0)$, for all ideals $I \subseteq R$ and all nonzero submodules $Y \subseteq X$.

Definition 2.1. The module ${}_RX$ is said to be M-prime if $\mathrm{Hom}_R(M, X) \neq 0$, and $\mathrm{Ann}_M(Y) = \mathrm{Ann}_M(X)$ for all submodules $Y \subseteq X$ such that $\mathrm{Hom}_R(M, Y) \neq 0$.

Proposition 2.2. The following conditions are equivalent for any left R-module X such that $\mathrm{Hom}_R(M, X) \neq 0$.

(1) X is an M-prime module;

(2) $N \cdot Y = (0)$ implies $N \cdot X = (0)$, for all submodules $N \subseteq M$ and all submodules $Y \subseteq X$ with $M \cdot Y \neq (0)$;

(3) $N \cdot Y = (0)$ implies $N \cdot X = (0)$, for all M-ideals $N \subseteq M$ and all nonzero M-generated submodules $Y \subseteq X$.

(5) "Universally" prime modules

In [Bican, Jambor, Kepka, Němec: 1980] a nonzero module ${}_RM$ is called prime if $\mathrm{rad}_N = \mathrm{rad}_M$ for all nonzero submodules $N \subseteq M$. This is equivalent to the condition that M is cogenerated by each of its nonzero submodules, and holds, for example, if M is a simple module, or if $M = R/P$, where P is a prime ideal of R.

Proposition 2.6. Each nonzero submodule of X cogenerates X iff X is M-prime for each module ${}_RM$ such that $\mathrm{Hom}_R(M, X) \neq 0$.

Example. Let $M = \mathbf{Z}_{p^\infty}$, in $\mathbf{Z}$–Mod. Then M is an M-prime module, but it is not a $\mathbf{Z}$-prime module, (and hence does not satisfy the stronger condition given above).

(6) When M itself is M-prime

Proposition 2.8. The module M is an M-prime module if and only if $f(M)$ cogenerates M, for each nonzero endomorphism $f \in \mathrm{End}_R(M)$.

${}_RX$ is *monoform* if each nonzero homomorphism $f : Y \to X$ defined on a submodule Y of X is a monomorphism.

Corollary 2.9. If M is monoform, then it is an M-prime module.

Example. Let $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ and $M = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, where F is a field. Then M is monoform, so it is an M-prime module.

Note that M is finitely generated and projective, with $\sigma[M] = R\text{–}Mod$.

(7) Prime M-ideals

Definition 3.1. The M-ideal P is said to be a *prime* M-ideal if there exists an M-prime module ${}_RX$ such that $P = \mathrm{Ann}_M(X)$.

Proposition 3.2. If P is a prime M-ideal, then $P = \mathrm{Ann}_M(X)$ for an M-generated M-prime module ${}_RX$.

Definition 3.3. Let P be a prime M-ideal. We say that P is a prime M-ideal associated to the module X if $P = \mathrm{Ann}_M(Y)$ for an M-prime submodule Y of X.

Proposition 3.4. Assume that M is a Noetherian module, and let X be any left R-module. If $\mathrm{Hom}_R(M, X) \neq 0$, then X has an associated prime M-ideal.

Definition 3.5. An M-ideal P is said to be a *primitive* M-ideal if $P = \mathrm{Ann}_M(S)$ for a simple module ${}_RS$.

Proposition 3.6. Maximal M-ideals are prime M-ideals; primitive M-ideals are prime M-ideals.

(8) In case M is projective in $\sigma[M]$.

For these results, assume M is projective in $\sigma[M]$. Note that $N \subseteq M$ is an M-ideal iff it is fully invariant in M.

Proposition 5.4. The following hold for any submodule $N \subseteq M$, and any submodule $Y \subseteq X$ in $\sigma[M]$,

(a) $N \cdot X = \sum_{f \in \mathrm{Hom}(M,X)} f(N)$

(b) $N \cdot (X/Y) = (0)$ if and only if $N \cdot X \subseteq Y$

Proposition 5.6. If K and N are submodules of M, then $(K \cdot N) \cdot X = K \cdot (N \cdot X)$ for any module ${}_RX$ in $\sigma[M]$.

Theorem 5.7. The following are equivalent for an M-ideal $P \subseteq M$.

(1) P is a prime M-ideal;

(2) $N \cdot K \subseteq P$ implies $N \subseteq P$ or $K \subseteq P$, for all M-ideals N and K such that K is M-generated;

(3) M/P is an M-prime module.

Proposition 5.9. If M is an M-prime module, then M is an R-prime module if $\mathrm{Hom}_R(M, N) \neq 0$ for all nonzero M-ideals N.

(9) In case M is Noetherian

Proposition 6.1. Let M be a Noetherian module, and assume that the module ${}_RX$ is a homomorphic image of a finite direct sum of copies of M. Then there exists a chain of submodules $(0) = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$ such that for $1 \leq i \leq n$, each factor module X_i/X_{i-1} is an M-prime module.

Definition 6.4. The module ${}_RX$ in $\sigma[M]$ is said to be finitely M-generated if there exists an epimorphism $f : M^n \to X$, for some positive integer n. It is said to be finitely M-annihilated if there exists a monomorphism $g : M/\operatorname{Ann}_M(X) \to X^m$, for some positive integer m.

Definition 6.5. The module ${}_RM$ is said to satisfy condition H if every finitely M-generated module is finitely M-annihilated.

Theorem 6.7. Let M be a Noetherian module. If M satisfies condition H and $\operatorname{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$, then there is a one-to-one correspondence between isomorphism classes of indecomposable M-injective modules and prime M-ideals.

(10) The analog of FBN rings

Proposition 6.8. Let M be a Noetherian module such that M satisfies condition H and $\text{Hom}_R(M, X) \neq 0$ for all modules X in $\sigma[M]$. If P is a prime M-ideal, then M/P is a semi-compressible module.

Lemma 6.9. Let M be a Noetherian module that satisfies condition H. If M is projective in $\sigma[M]$ and P is a maximal M-ideal, then M/P is a homogeneous semisimple module.

Proposition 6.10. Let M be a Noetherian module that satisfies condition H. If M is projective in $\sigma[M]$ and every prime M-ideal is maximal, then M has finite length.

Proposition 6.11. Let M be a module that satisfies condition H, and let N be an M-ideal. If M is projective in $\sigma[M]$, then M/N satisfies condition H.