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PERFECT QUOTIENT FUNCTORS

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Recently there has been considerable interest in extending to noncommutative rings the techniques of localization which have proved to be useful in the study of commutative rings. The notion of a quotient category, as utilized by Gabriel [5], has played a fundamental role in one approach to the problem. The constructions considered by Gabriel include quotient categories determined either by localizing Serre subcategories, certain filters of left ideals, or injective modules. (A special case of the latter method had been introduced by Findlay and Lambek [4].) Further equivalent notions are the torsion radicals of Maranda [12] (called idempotent kernel functors by Goldman [6]) and the (hereditary) torsion theories of Dickson [3] and Lambek [8]. The reader is also referred to the Walkers' paper [20] and the recent expositions of Lambek [9,11], Morita [14], and Stenström [18], which contain extensive bibliographies.

A major difficulty in this approach is that the quotient category in general need not coincide with the category of modules over the corresponding ring of quotients. This paper gives an expository account of the conditions under which the two categories coincide. It is intended to give the non-expert quick access to some of the basic definitions and theorems in noncommutative localization, and an exposure to some of the techniques of the area. It assumes a knowledge of elementary ring theory and category theory.

It seems appropriate to first review the relevant properties of localization in commutative rings. Let R be a commutative ring (all rings will be associative rings with identity element, and all modules will be unital), and let S be a multiplicative system in R (S is a multiplicatively closed subset of R with $0 \notin S$). The ring of quotients determined by S is a ring R_S and ring homomorphism $\alpha : R \to R_S$ with $\alpha(s)$ invertible for all $s \in S$, such that for each ring homomorphism $\eta : R \to R'$ with $\eta(s)$ invertible for all $s \in S$, there exists a unique homomorphism $\theta : R_S \to R'$ with $\eta = \theta\alpha$.

For a module ${}_RM$ the construction of a module of quotients M_S over R_S proceeds as follows. The first step is to factor out the submodule $\mathrm{rad}_S(M) = \{m \in M \mid sm = 0 \text{ for some } s \in S\}$, since no element of M_S can be annihilated by an invertible element of R_S. Note that $\mathrm{rad}_S(M/\mathrm{rad}_S(M)) = 0$. This construction defines a left exact subfunctor rad_S of the identity on the category of R-modules R–Mod, and motivates the general definition of a torsion radical, since it actually determines the module of quotients completely.

The module of quotients M_S is given by the set of pairs m/s, with $m \in M/\mathrm{rad}_S(M)$ and $s \in S$, with $m_1/s_1 = m_1/s_2$ if $s_2m_1 = s_1m_2$. If $\mathrm{rad}_S(M) = 0$, then for any element $0 \neq m/s \in M_S$, $0 \neq s(m/s) \in M$, so that each nonzero submodule of M_S has nonzero intersection with M, and thus M_S is an essential extension of M. This shows that if $\mathrm{rad}_S(M) = 0$, then M_S can be viewed as a submodule of the injective envelope $\mathrm{E}(M)$ of M (which is a maximal essential extension of M).

in fact, $M_S = \{x \in \mathrm{E}(M) \mid sx \in M \text{ for some } s \in S\}$, so that $M_S/M = \mathrm{rad}_S(\mathrm{E}(M)/M)$. In general, then, for a torsion radical σ, the module of quotients M_σ is constructed as the inverse image in $\mathrm{E}(M'')$ of $\mathrm{rad}_\sigma(\mathrm{E}(M'')/M'')$, where $M'' = M/\mathrm{rad}_\sigma(M)$.

The module of quotients M_S has the property that for each nonzero element $x \in M_S$ and each $s \in S$ there exists a unique element $y = x/s$ such that $sy = x$. In terms of homomorphisms, this states that for each $s \in S$, any R-homomorphism $f : Rs \to M_S$ can be extended uniquely to $g : R \to M_S$. For a torsion racial σ, this motivates the notion of a σ-torsionfree, σ-injective module, and the full subcategory $R\text{–}Mod/\sigma$ of σ-torsionfree, σ-injective R-modules is called the quotient category determined by σ.

The quotient functor $Q_\sigma : R\text{–}Mod \to R\text{–}Mod/\sigma$ is defined on modules by $Q_\sigma(M) = M_\sigma$. For a multiplicative system S in a commutative ring, M_S is naturally isomorphic to $R_S \otimes_R M$, and $R_S\text{–}Mod$ is equivalent to the quotient category $R\text{–}Mod/S$. Since the quotient functor is exact, R_S must be flat as an R-module, and since every element of R_S has the form $\alpha(R)/s$ for some $r \in R$ and $s \in S$, it can be shown that if $\beta : R_S \to R'$, $\gamma : R_S \to R'$ are ring homomorphisms with $\beta\alpha = \gamma\alpha$, then $\beta = \gamma$, so that α is an epimorphism in the category of rings. In general, the quotient functor Q_σ is naturally isomorphic to the functor $R_\sigma \otimes_R -$ if and only if the inclusion functor $R\text{–}Mod/\sigma \to R_\sigma\text{–}Mod$ is an equivalence, and in this case Q_σ is said to be a perfect quotient functor. Furthermore, there is a one-to-one correspondence between perfect quotient functors and flat epimorphic images of R. These localizations were studied by Silver [17]. If R is a Dedekind domain with field of quotients K, then every subring of K which contains F is a flat epimorphic extension of R, and so all such subrings are rings of quotients with perfect quotient functors. But every ring between R and K is a ring of quotients of R with respect to a multiplicative system if and only if the group of ideal classes of R is a torsion group ([6]), which shows that it is natural to study perfect quotient functors even for commutative rings.

Various properties of torsion radicals, quotient categories, and rings of quotients are summarized in Section 1. These give the necessary background for Section 2, in which several classes of objects in the quotient category are studied. The final section deals with perfect quotient functors. Some of the techniques utilized appear to be novel in that they make heavy use of the quotient category. Most of the proofs have been omitted in the first section, and have only been sketched in the last two sections.

§1. Some background results

A functor $\sigma : R\text{–}Mod \to R\text{–}Mod$ is called a *torsion radical* if for all modules ${}_RM$, ${}_RN$ and all $f \in \mathrm{Hom}_R(M, N)$, $\sigma M \subseteq M$, $f(\sigma M) \subseteq \sigma N$, $\sigma(M/\sigma M) = 0$ and $\sigma M' = \sigma M \cap M'$ for all submodules $M' \subseteq M$. The submodule $\sigma(M)$ will be denoted by $\mathrm{rad}_\sigma(M)$. A module ${}_RM$ is called *σ-torsion* if $\mathrm{rad}_\sigma(M) = M$, and *$\sigma$-torsionfree* if $\mathrm{rad}_\sigma(M) = 0$. Note that $\mathrm{rad}_\sigma(M)$ can be characterized either as the sum of σ-torsion submodules of M or as the intersection of submodules of M whose factor is σ-torsionfree.

The class of σ-torsion modules is closed under formation of epimorphic images, direct sums, (group) extensions, and submodules, and moreover, each such class is the torsion class of some torsion radical. Similarly, the class of σ-torsionfree modules is closed under submodules, direct products, (group) extensions, and essential extensions, and each such class is the torsionfree class of some torsion radical. Furthermore, ${}_RM$ is σ-torsion iff $\mathrm{Hom}_R(M, N) = 0$ for all σ-torsionfree modules ${}_RN$, and N is σ-torsionfree iff $\mathrm{Hom}_R(M, N) = 0$ for all σ-torsion modules M.

If σ, τ are torsion radicals such that $\mathrm{rad}_\sigma(M) \subseteq \mathrm{rad}_\tau(M)$ for all modules ${}_RM$, then the notation $\sigma \leq \tau$ is used, and τ is said to be larger than σ. If $W \in R\text{–}Mod$, then for any module ${}_RM$ let $\mathrm{rad}_W(M)$ be the intersection of all kernels of homomorphisms from M into W. The last statement of the following theorem is due to Jans [7], where the injective envelope of the direct product of a representative set of cyclic σ-torsionfree R-modules can be taken as W.

THEOREM (1.1). If ${}_RW$ is injective, then rad_W *defines a torsion radical of R–Mod, and $\sigma = \mathrm{rad}_{E(M)}$ is the largest torsion radical for which ${}_RM$ is σ-torsionfree. If σ is any torsion radical of R–Mod, then there exists an injective module ${}_RW$ such that $\sigma = \mathrm{rad}_W$.*

A torsion radical σ defines a closure operation on submodules $M' \subseteq M$ by letting $C_\sigma(M')$ be the inverse image in M of $\mathrm{rad}_\sigma(M/M')$. The submodule M' is called *σ-dense* if $C_\sigma(M') = M$, that is, if M/M' is σ-torsion, and *σ-closed* if $C_\sigma(M') = M'$, which occurs iff M/M' is σ-torsionfree. The set of σ-dense left ideals of R will be denoted by $\mathcal{D}_\sigma$. The closure of M' can be described as

$$C_\sigma(M') = \{m \in M \mid Dm \subseteq M' \text{ for some } D \in \mathcal{D}_\sigma\} ,$$

or if $\sigma = \mathrm{rad}_W$, as

$$\{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, N) \text{ such that } f(M') = 0\} .$$

It follows immediately that if $f \in \mathrm{Hom}_R(M, N)$ and $f(M') \subseteq N'$, then $f(C_\sigma(M')) \subseteq C_\sigma(N')$, and so if N is σ-torsionfree and $f, g \in \mathrm{Hom}_R(M, N)$ agree on M', then they must agree on $C_\sigma(M')$. For convenience, the σ-closure of M in its injective envelope $\mathrm{E}(M)$ will be denoted by $\mathrm{E}_\sigma(M)$.

The σ-dense submodules satisfy the following properties. If M_1 and M_2 are submodules of M, then (i) if $M_1 \subseteq M_2$ and M_1 is σ-dense, then M_2 is σ-dense; (ii) if M_1 and M_2 are σ-dense, then so is $M_1 \cap M_2$; (iii) M_1 is σ-dense iff $\{r \in R \mid rm \in M_1\} \in \mathcal{D}_\sigma$, for all $m \in M$; (iv) if $M_1 \subseteq M_2$ with M_1 σ-dense in M_2 and M_2 σ-dense in M, then M_1 is σ-dense in M. A nonempty set $\mathcal{D}$ of left ideals of R is called an *idempotent topologizing filter (IT-filter)* if the following conditions are satisfied: (i) $D \in \mathcal{D}$ and $D \subseteq A \subseteq R$ implies $A \in \mathcal{D}$; (ii) $D_1,\ D_2 \in \mathcal{D}$ implies $D_1 \cap D_2 \in \mathcal{D}$; (iii) $D \in \mathcal{D}$ implies $Dr^{-1} \in \mathcal{D}$ for all $r \in R$, where $Dr^{-1} = \{x \in R \mid xr \in D\}$; (iv) $D \in \mathcal{D}$, $A \subseteq D$, and $Ad^{-1} \in \mathcal{D}$ for all $d \in D$ implies $A \in \mathcal{D}$. The preceding remarks on σ-dense submodules make it clear that $\mathcal{D}_\sigma$ is an IT-filter. Conversely, if $\mathcal{D}$ is an IT-filter, then letting $\mathrm{rad}_{\mathcal{D}}(M) = \{m \in M \mid \mathrm{Ann}(m) \in \mathcal{D}\}$ defines a torsion radical. This can be used to establish the following theorem of Maranda [12], also implicit in Gabriel [5].

THEOREM (1.2). There is a one-to-one correspondence between torsion radicals of R–Mod and IT-filters of left ideals of R.

A module ${}_RM$ is called *σ-injective* if each homomorphism $f : N' \to M$ such that N' is a σ-dense submodule of ${}_RN$ can be extended to N. This reduces to the usual definition of injectivity if σ is the identity functor. It can be shown in the standard way that a direct product of modules if σ-injective iff each factor is σ-injective. It is almost clear that M is σ-injective iff $M = \mathrm{E}_\sigma(M)$, and in fact $\mathrm{E}_\sigma(M)$ defines a "σ-injective envelope" of M. Baer's condition holds in this situation, in that M is σ-injective iff each homomorphism $f : D \to M$ such that $D \in \mathcal{D}_\sigma$ can be extended to R.

PROPOSITION (1.3). The following conditions are equivalent for $M \in R\text{–}Mod$:

(a). M *is* σ*-torsionfree and* σ*-injective.*

(b). Each homomorphism $f : N' \to M$ *such that* N' *is a* σ*-dense submodule of* ${}_R N$ *can be extended uniquely to* N.

(c). $\mathrm{Hom}_R(N, M) = 0 = \mathrm{Ext}^1(N, M)$ *for all* σ*-torsion modules* N.

(d). $\mathrm{E}(M)$ *and* $\mathrm{E}(M)/M$ *are* σ*-torsionfree.*

The full subcategory of $R\text{–}Mod$ determined by all σ-torsionfree, σ-injective modules will be denoted by $R\text{–}Mod/\sigma$. If $f \in \mathrm{Hom}_R(M, N)$ and M, $N \in R\text{–}Mod/\sigma$, then $\ker(f)$ is σ-closed in M. Since any submodule of a σ-torsionfree module is σ-torsionfree, and a σ-closed submodule of a σ-injective module is σ-injective, $\ker(f) \in R\text{–}Mod/\sigma$. On the other hand if f is a monomorphism, then $N/f(M)$ is σ-torsionfree, since a σ-injective submodule of a σ-torsionfree module is σ-closed, and so M is the kernel of the natural homomorphism $N \to \mathrm{E}_\sigma(N/f(M))$. This shows that the category $R\text{–}Mod/\sigma$ has kernels, and that every monomorphism is a kernel. Since it clearly has finite direct sums and a zero object, to show that $R\text{–}Mod/\sigma$ is an abelian category it is sufficient to show that it has cokernels and that every epimorphism is a cokernel.

It is convenient to define the *quotient functor* Q_σ at this point. Let $Q_\sigma(M) = \mathrm{E}_\sigma(M/\mathrm{rad}_\sigma(M))$, for $M \in R\text{–}Mod$. Then if ${}_R N \in R\text{–}Mod/\sigma$ and $f : M \to N$, f factors through $M/\mathrm{rad}_\sigma(M)$ since $f(\mathrm{rad}_\sigma(M)) \subseteq \mathrm{rad}_\sigma(N) = 0$. Since N is σ-torsionfree and σ-injective, by Proposition 1.3 there is a unique extension of f to $\mathrm{E}_\sigma(M/\mathrm{rad}_\sigma(M))$ since $M/\mathrm{rad}_\sigma(M)$ is a σ-dense submodule of $\mathrm{E}_\sigma(M/\mathrm{rad}_\sigma(M))$. This shows how to define the functor Q_σ on homomorphisms, and more importantly shows that Q_σ is a left adjoint of the inclusion $U_\sigma : R\text{–}Mod/\sigma \to R\text{–}Mod$.

If $f\colon M \to N$ is a homomorphism in R–Mod/σ, then $Q_\sigma(\mathrm{coker}(f))$ serves as the cokernel of f in R–Mod/σ. Furthermore, if f is an epimorphism in R–Mod/σ, then $Q_\sigma(\mathrm{coker}(f)) = 0$ and so $f(M)$ must be σ-dense in N, which shows that N is the cokernel of the inclusion $\ker(f) \to M$. Thus R–Mod/σ is an abelian category, and so Q_σ is right exact since it is a left adjoint. (It is easy to show that Q_σ preserves monomorphisms, so in fact Q_σ is exact. It is important to note that the inclusion functor U_σ need not be exact, since an epimorphism in R–Mod/σ may not be an epimorphism in R–Mod, and in fact $U_\sigma Q_\sigma$ need not be exact.) The above remarks can be used to establish the following important theorem. (See Gabriel [5] or Mitchell [13].)

THEOREM (1.4) Let σ *be a torsion radical of* R–Mod. *Then the quotient category* R–Mod/σ *is an abelian category, and the inclusion function* $U_\sigma : R\text{–}Mod/\sigma \to R\text{–}Mod$ *has an exact left adjoint* $Q_\sigma : R\text{–}Mod \to R\text{–}Mod/\sigma$.

As shown by Gabriel [5], an alternate method of constructing $Q_\sigma(M)$ is to take the direct limit of $\mathrm{Hom}_R(D, M/\mathrm{rad}_\sigma(M))$ over all $D \in \mathcal{D}_\sigma$. Theorem 1.4 has a converse, in that any full abelian subcategory of R–Mod whose inclusion functor has an exact left adjoint is equivalent to a subcategory of the form R–Mod/σ for some torsion radical σ. Another useful characterization of subcategories of this form is given by a proposition due to Morita [14].

PROPOSITION (1.5) Let $\mathcal{A}$ *be a full subcategory of* R–Mod. *Then* $\mathcal{A} = R\text{–}Mod/\sigma$ *for some torsion radical* $\sigma \iff \mathcal{A}$ *is closed under isomorphisms, direct summands, direct products, and* $M \in \mathcal{A}$ *iff* $\mathrm{E}(M)$, $\mathrm{E}(\mathrm{E}(M)/M) \in \mathcal{A}$.

The *module of quotients* $Q_\sigma(M)$ will be denoted simply by M_σ. For any module $M \in R\text{–}Mod/\sigma$, and any element $m \in M$, the homomorphism $[r \mapsto rm] : R \to M$ defined by multiplication can be extended uniquely to $\rho_m : R_\sigma \to M$. For any element $q \in R_\sigma$, ρ_q can be used to define right multiplication by q, and this induces a ring structure on R_σ. Furthermore, any module $M \in R\text{–}Mod/\sigma$ becomes a left R_σ-module by defining $qm = \rho_m(q)$, for all $q \in R$ and $m \in M$. In fact, a σ-torsionfree R-module can be given an R_σ-module structure iff it is generated in the categorical sense by ${}_RR_\sigma$ ([2]). The ring R_σ is called the *ring of quotients determined by* σ.

THEOREM (1.6). The module R_σ has a ring structure, and every R-module in $R\text{–}Mod/\sigma$ is an R_σ-module. Every R-homomorphism in $R\text{–}Mod/\sigma$ is an R_σ-homomorphism.

A module $M \in R\text{–}Mod/\sigma$ is injective as an R_σ-module iff it is injective as an R-module, and so it follows from Proposition 1.5 that $R\text{–}Mod/\sigma$ is a quotient category of $R_\sigma\text{–}Mod$ as well as of $R\text{–}Mod$. It can be shown that there exists an injective module ${}_RW$ with $\sigma = \mathrm{rad}_W$ such that $R/\operatorname{Ann}(W)$ can be embedded in W, and that R_σ is isomorphic to the bicommutator (double centralizer) of W. More generally, note that R_σ is isomorphic to the bicommutator of W provided only that W is finitely generated over its endomorphism ring, so this gives another method of constructing rings of quotients.

The ring of quotients $Q_{\max}(R)$ determined by $\mathrm{rad}_{E(R)}$ is called the *complete ring of left quotients* of R. A torsion radical σ will be called *complete* if $\sigma = \mathrm{rad}_{E(R/K)}$ for $K = \mathrm{rad}_\sigma(R)$. In this case, since $\mathrm{E}(R/K)$ is annihilated by K, it is the R/K-injective envelope of R/K, and so $R_\sigma = Q_{\max}(R/K)$.

§2. Objects in the quotient category

In the remaining sections σ will be a fixed torsion radical. The quotient category $R\text{–}Mod/\sigma$ has direct products and a generator, R_σ. The categorical direct sum $\coprod_{\alpha \in I} M_\alpha$ of a set of modules $\{M_\alpha\}_{\alpha \in I}$ in $R\text{–}Mod/\sigma$ is given by $\mathrm{E}_\sigma(\oplus_{\alpha \in I} M_\alpha)$. The next propositions investigate objects in $R\text{–}Mod/\sigma$ which are injective, projective, and have various chain conditions or finiteness conditions.

PROPOSITION (2.1). Let $M, W \in R\text{–}Mod$, with ${}_RW$ injective.

(a). There is a one-to-one correspondence between subobjects of M_σ in $R\text{–}Mod/\sigma$ and σ-closed submodules of M.

(b). If $M \in R\text{–}Mod/\sigma$, then M is injective (quasi-injective) in $R\text{–}Mod/\sigma \iff M$ is injective (quasi-injective) in $R\text{–}Mod$.

(c). W is an injective cogenerator in $R\text{–}Mod/\sigma \iff \mathrm{rad}_W = \sigma$.

Proof. (a). Since every σ-closed submodule of M contains $\mathrm{rad}_\sigma(M)$, σ-closed submodules of $M/\mathrm{rad}_\sigma(M)$ correspond to those of M. If M is σ-torsionfree, then there is a one-to-one correspondence between σ-closed submodules of M and $\mathrm{E}_\sigma(M)$, where a submodule is extended by taking its σ-closure in $\mathrm{E}_\sigma(M)$ and contracted by intersection. Finally, the subobjects of a module in $R\text{–}Mod/\sigma$ are just its σ-closed submodules.

(b). The inclusion functor U_σ preserves injectives since it has an exact left adjoint, and if ${}_RM$ is injective, then it is obviously injective in $R\text{–}Mod/\sigma$. This shows that injective envelopes are the same in both categories, and so the statement on quasi-injectives follows from the characterization of quasi-injective objects as fully invariant subobjects of their injective envelopes.

(c). W is a cogenerator in R–Mod/σ iff each module in R–Mod/σ can be embedded in a direct product of copies of W, and this occurs iff $\mathrm{rad}_W = \sigma$.

PROPOSITION (2.2). If $\sigma = \mathrm{rad}_{E(X)}$ *for* $X \in R$–Mod *and* $\mathrm{rad}_\sigma(R) = K$*, then the following conditions are equivalent.*

(a). Every object in R–Mod/σ *is injective.*

(b). Every essential left ideal of R/K *is* σ*-dense in* R/K*.*

(c). X *is a nonsingular* R/K*-module.*

Proof. (a)⇒(b). If A is an essential left ideal of R/K, then $\mathrm{E}(R/K) = \mathrm{E}(A)$, and if every object in R–Mod/σ is injective, then $Q_\sigma(R/K) = Q_\sigma(A)$, so A is σ-dense in R/K.

(b) ⇒ (c). Recall that X is nonsingular if no nonzero element has an essential annihilator. Since $\sigma = \mathrm{rad}_{E(X)}$, the annihilator of any element of X is σ-closed in R/K, so by assumption it cannot be essential.

(c) ⇒ (a). If $M \in R$–Mod/σ, then $\mathrm{E}(M)$ and $\mathrm{E}(M)/M$ are σ-torsionfree, and so they are R/K-modules. The annihilator in R/K of any element of $\mathrm{E}(M)/M$ is essential and σ-closed in R/K, and it can also be shown to be σ-dense since $\sigma = \mathrm{rad}_{E(X)}$ and X is nonsingular over R/K. This is a contradiction, so $M = \mathrm{E}(M)$, and thus M is injective in R–Mod/σ.

The above proposition is a slight modification of a result of Lambek [10]. Note that in this case R_σ is von Neumann regular since every principal left ideal belongs to R–Mod/σ and hence is a direct summand of R_σ.

By Proposition 2.1, for ${}_RM$ the module of quotients M_σ is Noetherian (Artinian) in R–Mod/σ iff the σ-closed submodules of M satisfy the ascending chain condition (d.c.c., respectively). A module $M \in R$–Mod/σ

will be called *finitely generated* (in R–Mod/σ) if for any epimorphism $\coprod_{\alpha \in I} M_\alpha \to M$ in R–Mod/σ there is a finite subset $F \subseteq I$ such that the natural morphism $\coprod_{\alpha \in F} M_\alpha \to \coprod_{\alpha \in I} M_\alpha \to M$ is an epimorphism. The next proposition can be used to show that for ${}_RM$, M_σ is Noetherian iff every subobject is finitely generated (in R–Mod/σ).

PROPOSITION (2.3). Let $M \in R$–Mod. Then M_σ is finitely generated in R–Mod/σ $\Longleftrightarrow$ every σ-dense submodule of M contains an R-finitely generated σ-dense submodule.

Proof. $\Rightarrow$). If M_σ is finitely generated in R–Mod/σ, and M' is a σ-dense submodule, then M' is a quotient of a free module R^I. Applying Q_σ yields $\coprod_I Q_\sigma(R) \to Q_\sigma(M') = M_\sigma$, an epimorphism, since Q_σ is exact and preserves direct sums. Since M_σ is finitely generated, for some finite subset F the image of $R^F \to M'$ is σ-dense in M'.

$\Leftarrow$). If the condition holds for M, then it is inherited by $M/\mathrm{rad}_\sigma(M)$ and then by $\mathrm{E}_\sigma(M/\mathrm{rad}_\sigma(M)) = M_\sigma$. The conclusion then follows from the fact that a module in R–Mod/σ is finitely generated there iff every σ-dense sum of σ-closed submodules has a finite subcollection whose sum is σ-dense.

COROLLARY (2.4). Let $X \in R$–Mod, $\sigma = \mathrm{rad}_{E(X)}$, and $K = \mathrm{rad}_\sigma(R)$. If X is nonsingular as an R/K-module, then X_σ is finitely generated in R–Mod/σ $\Longleftrightarrow$ ${}_RX$ has finite uniform dimension.

Proof. If X is a nonsingular R/K-module, then its σ-dense submodules coincide with its essential submodules, and so every essential submodule contains a finitely generated essential submodule iff ${}_RX$ has finite uniform dimension (that is, X does not contain an infinite direct sum of nonzero submodules).

A module $M \in R\text{–}Mod/\sigma$ is called small if $\mathrm{Hom}_R(M, -)$ preserves direct sums. This occurs iff each homomorphism $f : M \to \coprod_{\alpha \in I} M_\alpha$ admits a factorization through $\coprod_{\alpha \in I} M_\alpha$ for some finite subset $F \subseteq I$. This motivates the next definition. A module $M \in R\text{–}Mod$ will be called *σ-small* if for each homomorphism $f : M \to \oplus_{\alpha \in I} M_\alpha$ such that $M_\alpha \in R\text{–}Mod/\sigma$ for all $\alpha \in I$, f can be factored through the direct sum of a finite subcollection. The next proposition follows rather quickly from the definition.

PROPOSITION (2.5). Let $M \in R\text{–}Mod$. Then M_σ is small in $R\text{–}Mod/\sigma \iff$ every σ-dense submodule of M is σ-small.

PROPOSITION (2.6). Let $D \in \mathcal{D}_\sigma$. Then ${}_RD$ is σ-small $\iff$ for each ascending chain $\{A_i\}_{i=1}^\infty$ of left ideals with $\cup_{i=1}^\infty A_i = D$, $A_i \in D_\sigma$ for some i.

Proof. $\Rightarrow$). If $D \in \mathcal{D}_\sigma$ and $D = \cup_{i=1}^\infty A_i$, then the natural homomorphism $D \to \oplus_{i=1}^\infty Q_\sigma(R/A_i)$ factors through a finite direct sum, and so $Q_\sigma(R/A_i) = 0$ for some i, which shows that A_i is σ-dense.

$\Leftarrow$). Given a homomorphism $g : D \to \oplus_{\alpha \in I} M_\alpha$, extend g to $f : R \to \prod_{\alpha \in I} M_\alpha$. Let F be the set of indices α such that $f(1)_\alpha \neq 0$. For any countable subset of F let $A_i = \{d \in D \mid f_n(d) = 0 \text{ for } n \geq i\}$. For any $d \in D$, $f_n(d) = 0$ for all but finitely many components, so $\cup_{i=1}^\infty A_i = D$, and therefore by assumption A_i is σ-dense for some i. But now $A_i f_n(1) = 0$ for all $n \geq i$, and this implies $f_n(1) = 0$ for all $n \geq i$ since each M_n is σ-torsionfree. This shows that every countable subset of F must be finite, and hence F must be finite, so that g factors through the direct sum of a finite subcollection of modules M_α, and D is σ-small.

A module ${}_RP$ will be called *σ-projective* if each homomorphism $f : P \to N$ such that $g : M \to N$ is an epimorphism, with $M \in R\text{–}Mod/\sigma$ and N σ-torsionfree, can be lifted to $h : P \to N$ with $gh = f$. This is equivalent to the condition that each homomorphism $f : P \to N$ such that $g : M \to N$ is an epimorphism, with M, N σ-torsionfree, can be lifted to a σ-dense submodule P' of P. (The latter is the definition used by Goldman [6].)

PROPOSITION (2.7). Let $P \in R\text{–}Mod$. Then P_σ is projective in $R\text{–}Mod/\sigma \iff$ every σ-dense submodule of P is σ-projective.

Proof. $\Rightarrow$). If P' is σ-dense in P and $f : P' \to N$ with $g : M \to N$ an epimorphism, then f extends uniquely to $f_\sigma : Q_\sigma(P') \to Q_\sigma(N)$. If $M \in R\text{–}Mod/\sigma$ and N is σ-torsionfree, then $M \to Q_\sigma(N)$ is an epimorphism in $R\text{–}Mod/\sigma$, and so f_σ lifts to M because $Q_\sigma(P')$ is isomorphic to P_σ and therefore projective in $R\text{–}Mod/\sigma$.

$\Leftarrow$). If $g : M \to N$ is an epimorphism in $R\text{–}Mod/\sigma$, and $f : P_\sigma \to N$, then the inverse image $P' \subseteq P$ of $g(M)$ is σ-dense in P since $g(M)$ is σ-dense in N. By assumption the homomorphism $P' \to g(M)$ can be lifted to M, and then it can be extended uniquely to P_σ, and the extension to P_σ is the required lifting of f.

A module ${}_RM$ is called *monoform* if every nonzero homomorphism from a submodule of M into M is a monomorphism. A torsion radical σ is said to be a *prime torsion radical* ([6]) if $\sigma = \mathrm{rad}_{E(M)}$ with M monoform. The next proposition gives several characterizations of monoform modules, and shows that a torsion radical σ is a prime torsion radical iff the quotient category $R\text{–}Mod/\sigma$ has an injective cogenerator which is the injective envelope of a simple object.

PROPOSITION (2.8). Let $M \in R\text{–}Mod$, and let $\sigma = \mathrm{rad}_{E(M)}$. Then the following conditions are equivalent:

(a). M is monoform.

(b). M_σ is simple in $R\text{–}Mod/\sigma$.

(c). M_σ is quasi-injective in $R\text{–}Mod$, and $\mathrm{End}_R(M_\sigma)$ is a division ring.

(d). $\mathrm{End}_R(\overline{M})$ is a division ring, where $\overline{M}$ is the quasi-injective envelope of ${}_RM$.

Proof. (a)⇒(b). If M is monoform, then $\mathrm{Hom}_R(M_2/M_1, M) = 0$ for all submodules $0 \neq M_1 \subseteq M_2 \subseteq M$, or equivalently, $\mathrm{Hom}_R(M/M_1, \mathrm{E}(M)) = 0$ for all $0 \neq M_1 \subseteq M$. Thus every nonzero submodule of M is σ-dense, and so M_σ has no nontrivial subobjects in $R\text{–}Mod/\sigma$.

(b)⇒(c). If M_σ is simple, then its endomorphism ring in $R\text{–}Mod/\sigma$, which is just $\mathrm{End}_R(M_\sigma)$, must be a division ring, and furthermore, it must be quasi-injective in $R\text{-}Mod$ since it is quasi-injective in $R\text{-}Mod/\sigma$.

(c) ⇒ (d). If M_σ is quasi-injective, then $M \subseteq \overline{M} \subseteq M_\sigma$, so each endomorphism of $\overline{M}$ can be extended uniquely to an endomorphism of M_σ. Since $\overline{M}$ is invariant in M_σ, $\mathrm{End}_R(\overline{M}) = \mathrm{End}_R(M_\sigma)$.

(d) ⇒ (a). Every nonzero homomorphism from a submodule of M into M extends to an endomorphism of $\overline{M}$, and if $\mathrm{End}_R(\overline{M})$ is a division ring, then this extension must be a monomorphism.

The various conditions of the above proposition are found in [6], [15] and [19]. These papers also study a primary decomposition for Noetherian modules over an arbitrary ring, in which the role of prime ideals is played by the prime torsion radicals. A module ${}_RM$ is called *rationally complete* if $M = M_\sigma$ for $\sigma = \mathrm{rad}_{E(M)}$. In the above proposition M_σ is rationally complete and monoform, and $\mathrm{E}(M_\sigma)$ also defines σ. Since the quotient category $R\text{–}Mod/\sigma$ has an injective cogenerator which is the injective envelope of a simple object, it has a unique simple object, up to isomorphism. This shows that there is a one-to-one correspondence between prime torsion radicals and isomorphism classes of rationally complete monoform modules.

§3. Perfect quotient functors

THEOREM (3.1). The following conditions are equivalent for a torsion radical σ:

(a). The inclusion $V_\sigma : R\text{–}Mod/\sigma \to R_\sigma\text{–}Mod$ is an equivalence.

(b). $V_\sigma Q_\sigma$ is naturally isomorphic to $R_\sigma \otimes_R -$.

(c). The inclusion $U_\sigma : R\text{–}Mod/\sigma \to R\text{–}Mod$ is exact and preserves direct sums.

(d). Every R_σ-module is σ-torsionfree.

(e). $R_\sigma D = R_\sigma$ for each $D \in \mathcal{D}_\sigma$.

(f). R_σ is a flat right R-module, the canonical ring homomorphism $R \to R_\sigma$ is an epimorphism in the category of rings, and the canonical homomorphism $R_\sigma \otimes_R D \to R$ is an isomorphism for each $D \in \mathcal{D}_\sigma$.

Proof. A circular argument will be used to show that (a) - (d) are equivalent, and then it will be shown that (a), (b) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (d). The diagram at the right illustrates the situation.

$$\begin{tikzcd}
R\text{–}Mod \arrow[d, shift right, "R_\sigma \otimes -"'] \arrow[dd, bend right=60, "Q_\sigma"'] \\
R_\sigma\text{–}Mod \arrow[u, shift right] \\
R\text{–}Mod/\sigma \arrow[u, "V_\sigma"'] \arrow[uu, bend right=60, "U_\sigma"']
\end{tikzcd}$$

(a) $\Rightarrow$ (b). If V_σ is an equivalence, then $V_\sigma Q_\sigma$ is a left adjoint for the inclusion $R_\sigma\text{–}Mod \to R\text{–}Mod$ since Q_σ is a left adjoint for U_σ, and so $V_\sigma Q_\sigma$ is naturally isomorphic to $R_\sigma \otimes_R -$ by the uniqueness of adjoint functors.

(b) $\Rightarrow$ (c). If $V_\sigma Q_\sigma \simeq R_\sigma \otimes_R -$, then $V_\sigma Q_\sigma$ must be right exact and preserve direct sums, and from this it can be shown that V_σ, and hence U_σ, is exact and preserves direct sums.

(c) $\Rightarrow$ (d). For $M \in R\text{–}Mod/\sigma$, let $F_1 \xrightarrow{g} F_0 \xrightarrow{f} M \to 0$ be a free resolution of M in $R_\sigma\text{–}Mod$. If U_σ preserves direct sums, then $F_1, F_0 \in R\text{–}Mod/\sigma$, and if U_σ is right exact, then $g : F_1 \to \mathrm{E}_\sigma(\mathrm{Im}(g))$ must be epic in $R_\sigma\text{–}Mod$ since it is epic in $R\text{–}Mod/\sigma$. Thus $\ker(f) = \mathrm{Im}(g) \in R\text{–}Mod/\sigma$, and M is σ-torsionfree.

(d) $\Rightarrow$ (a). For $M \in R_\sigma\text{–}Mod$, $\mathrm{E}_\sigma(M) \in R\text{–}Mod/\sigma$ if M is σ-torsionfree, and so $\mathrm{E}_\sigma(M)/M$ is an R_σ-module which is by assumption σ-torsionfree. This implies that $M = \mathrm{E}_\sigma(M) \in R\text{–}Mod/\sigma$.

(a), (b) $\Rightarrow$ (f). If (a) and (b) hold, then $R_\sigma \otimes_R -$ is exact, and so $(R_\sigma)_R$ is flat. The inclusion $R_\sigma\text{–}Mod \to R\text{–}Mod$ must be a full functor since V_σ is an equivalence and U_σ is full, and this shows that $R \to R_\sigma$ is an epimorphism in the category of rings. (In fact, by [9] the conditions are equivalent.) If $D \in \mathcal{D}_\sigma$, then $Q_\sigma(R/D) = 0$, and so $V_\sigma Q_\sigma(D) \to V_\sigma Q_\sigma(R)$ is an isomorphism, which implies by (b) that $R_\sigma \otimes_R D \to R_\sigma \otimes_R R \to R_\sigma$ is an isomorphism.

(f) $\Rightarrow$ (e). $R_\sigma D$ is the image in R_σ of $R_\sigma \otimes_R D \to R$.

(e) $\Rightarrow$ (d). If $M \in R_\sigma\text{–}Mod$, and $Dm = 0$ for some $m \in M$ and $D \in \mathcal{D}_\sigma$, then $R_\sigma m = R_\sigma D m = 0$ implies $m = 0$, and thus M is σ-torsionfree.

The torsion radical σ or the quotient functor Q_σ is said to be *perfect* ([18]) if the conditions of the theorem are satisfied. It seems logical to say that Q_σ is *hereditary* if it preserves projective objects, and *Noetherian* if it preserves finitely generated objects.

Certain parts of Theorem 3.1 are proved by Gabriel [5] and Maranda [12], and essentially all of the conditions are studied by the Walkers

[20]. The next few propositions are for the most part well-known. Popescu and Spircu [16] give a converse to condition (f), in that any right flat epimorphic image of R is a ring of quotients. In particular, if $R \to R'$ is an epimorphism in the category of rings and R'_R is flat, then R'–Mod is a full abelian subcategory of R–Mod whose inclusion functor has an exact left adjoint $R' \otimes_R -$, so it is a quotient category of R–Mod, and R' is isomorphic to the ring of quotients determined by a perfect torsion radical. This leads to the following proposition [18].

PROPOSITION (3.2). There is a one-to-one correspondence between equivalence classes of ring epimorphisms $R \to R'$ such that R'_R is flat, and perfect torsion radicals of R–Mod.

The functor U_σ is naturally isomorphic to $\mathrm{Hom}_R(R_\sigma, -)$, and so U_σ is exact iff R_σ is projective in R–Mod/σ and U_σ preserves direct sums iff R_σ is small in R–Mod/σ. Most parts of the proofs of the following three propositions follow immediately from this remark and Propositions 2.3, 2.5, 2.6 and 2.7.

PROPOSITION (3.3). The following conditions are equivalent:

(a). Q_σ is hereditary.

(b). U_σ is exact.

(c). R_σ is projective in R–Mod/σ.

(d). Each σ-dense left ideal of R contains a σ-projective, σ-dense left ideal.

Proof. Since Q_σ is a left adjoint for U_σ, it follow that (b) $\Rightarrow$ (a). It is clear that (a) $\Rightarrow$ (c) $\Rightarrow$ (b) and (c) $\Leftrightarrow$ (d).

PROPOSITION (3.4). The following conditions are equivalent:

(a). Q_σ is Noetherian.

(b). R_σ is finitely generated in R–Mod/σ.

(c). Each σ-dense left ideal of R contains a finitely generated σ-dense left ideal.

Proof. Clearly (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). Assume that (c) holds, and that $M \in R$–Mod is finitely generated, say $M = \sum_{i=1}^{n} Rm_i$, with $m_i \in M$. If $M' \subseteq M$ is σ-dense, then $A_i = \{r \in R \mid rm_i \in M\}$ is σ-dense, so it contains a finitely generated left ideal $D_i \in \mathcal{D}_\sigma$, and so $\sum_{i=1}^{n} D_i m_i$ is a finitely generated σ-dense submodule of M'. This shows that $Q_\sigma(M)$ is finitely generated.

PROPOSITION (3.5). The following conditions are equivalent:

(a). Q_σ preserves small objects.

(b). U_σ preserves direct sums.

(c). R_σ is small in R–Mod/σ.

(d). If $\{A_i\}_{i=1}^{\infty}$ is an ascending chain of left ideals such that $\cup_{i=1}^{\infty} A_i \in \mathcal{D}_\sigma$, then $A_i \in \mathcal{D}_\sigma$ for some i.

Proof. Clearly (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), (c) $\Leftrightarrow$ (d), and (b) $\Rightarrow$ (a) follows easily from the fact that Q_σ is a left adjoint for U_σ.

If R is left hereditary and left Noetherian, then the quotient functor Q_σ must be hereditary and Noetherian. In fact, using the conditions on R_σ as an object in R–Mod/σ viewed as a quotient category of R_σ–Mod shows that Q_σ is hereditary if the ring R_σ is left hereditary, and that Q_σ is Noetherian if the ring R_σ is left Noetherian. If Q_σ is Noetherian, then it must preserve small objects, and so the following proposition holds.

PROPOSITION (3.6). The following conditions are equivalent:

(a). Q_σ is perfect.

(b). Q_σ is hereditary and Noetherian.

(c). Q_σ is a small projective object in $R\text{–}Mod/\sigma$.

PROPOSITION (3.7). Let $W \in R\text{–}Mod$ be injective, with $\sigma = \mathrm{rad}_W$.

(a). Q_σ is perfect $\Longleftrightarrow$ W is a cogenerator in $R_\sigma\text{–}Mod$.

(b). If σ is complete, then Q_σ is perfect $\Longleftrightarrow$ R_σ contains a copy of each simple R_σ-module.

Proof. (a). If Q_σ is perfect, then $R_\sigma\text{–}Mod$ is equivalent to $R\text{–}Mod/\sigma$, and W is a cogenerator in $R\text{–}Mod/\sigma$ since $\sigma = \mathrm{rad}_W$. If W is a cogenerator in $R_\sigma\text{–}Mod$, then every R_σ -module is σ-torsionfree and hence Q_σ is perfect.

(b). If σ is complete, then $\sigma = \mathrm{rad}_W$ for $W = \mathrm{E}(R/\mathrm{rad}_\sigma(R)) = \mathrm{E}(R_\sigma)$, and $\mathrm{E}(R_\sigma)$ is a cogenerator in $R_\sigma\text{–}Mod$ iff it contains a copy of each simple R_σ-module.

The next proposition characterizes perfect prime torsion radicals. A commutative ring has a unique maximal ideal iff it has a unique simple module (up to isomorphism), and this motivates the definition of a prime torsion radical as one for which the quotient category has a unique simple object (up to isomorphism). To obtain any useful conditions on the ring of quotients it appears to be necessary to assume that the torsion radical is perfect.

PROPOSITION (3.8). Let $M \in R\text{–}Mod$, with M monoform and rationally complete, and let $\sigma = \mathrm{rad}_{E(M)}$.

(a). σ is a perfect prime $\Longleftrightarrow$ $R_\sigma/\mathrm{J}(R_\sigma)$ is a simple ring and M is the only simple R_σ-module.

(b). If σ is a perfect prime, then $R_\sigma/\mathrm{J}(R_\sigma)$ is simple Artinian $\Longleftrightarrow$ M is finitely generated over its endomorphism ring.

Proof. Since M is rationally complete, $M = M_\sigma$. The Jacobson radical of R_σ is denoted by $\mathrm{J}(R_\sigma)$.

(a). If σ is perfect, then $R\text{–}Mod/\sigma$ is equivalent to $R_\sigma\text{–}Mod$. This shows that if σ is a prime, then M must be the unique simple R_σ-module (up to isomorphism), and so $\mathrm{J}(R_\sigma) = \mathrm{Ann}(M)$ is a maximal ideal of R_σ.

Conversely, if $R_\sigma/\mathrm{J}(R_\sigma)$ is simple, R_σ has only one isomorphism class of simple modules, and M is simple, then $\mathrm{E}(M)$ is a cogenerator in $R_\sigma\text{–}Mod$. Therefore σ is perfect, and it must be a prime since M is simple in $R\text{–}Mod/\sigma$ and $\mathrm{End}_R(M)$ is a division ring.

(b). If M is finitely generated over $\mathrm{End}_R(M)$, then $R_\sigma/\mathrm{J}(R_\sigma)$ can be embedded in a finite direct sum of copies of M since $\mathrm{J}(R_\sigma) = \mathrm{Ann}(M)$. This shows that $R_\sigma/\mathrm{J}(R_\sigma)$ is simple Artinian. The converse is clear.

PROPOSITION (3.9). If σ is a complete torsion radical, with $\mathrm{rad}_\sigma(R) = K$, then the following conditions are equivalent:

(a). R_σ is semisimple Artinian.

(b). Q_σ is perfect and R/K is a nonsingular left R/K-module.

(a). The module ${}_{R/K}R/K$ is nonsingular and has finite uniform dimension.

Proof. (a) $\Leftrightarrow$ (b). If R_σ is semisimple Artinian, then R_σ is a cogenerator in $R_\sigma\text{–}Mod$. Since σ is complete, this implies that Q_σ is perfect, and then every object in $R\text{–}Mod/\sigma$ is injective since $R\text{–}Mod/\sigma$ is equivalent to $R_\sigma\text{–}Mod$. By Proposition 2.2, ${}_{R/K}R/K$ is nonsingular.

Conversely, if Q_σ is perfect and ${}_{R/K}R/K$ is nonsingular, then R_σ–Mod and R–Mod/σ are equivalent and Proposition 2.2 shows that R_σ is semisimple Artinian, since every left R_σ-module is injective.

(b)⇔(c). If ${}_{R/K}R/K$ is nonsingular, then every object in R–Mod/σ is injective, hence projective, and so Q_σ is hereditary. Thus Q_σ is perfect iff R_σ is finitely generated in R–Mod/σ, and the proof can be completed by applying Corollary 2.4.

The above result is contained in Gabriel [5], since σ is complete and hence $R_\sigma = Q_{\max}(R/K)$. Note that as a consequence of Proposition 3.8, in this situation R_σ is simple iff σ is a perfect prime.

The final theorem presents a proof (from [2]) of a variant of Goldie's theorem characterizing orders in semisimple Artinian rings. The proof uses Propositions 2.8 and 3.9, and does not appeal to the Artin-Wedderburn theorem describing semisimple Artinian rings (which, in fact, it contains as a special case).

Several preliminary notes are necessary before proving the theorem. If ${}_RU$ is uniform and ${}_RX$ is nonsingular, then it is easy to show that any nonzero homomorphism from U to X must be a monomorphism. In particular, a nonsingular uniform module must be monoform. The ring R has finite uniform dimension, with $\dim(R) = n$, iff there is an essential direct sum $U_1 \oplus \cdots \oplus U_n$ of uniform left ideals. In this case, if $0 \to R \to U^m$ is an exact sequence, then it can be shown that for some $k \leq n$ there is an exact sequence $0 \to R \to U^k$.

THEOREM (3.10). If R is semiprime and ${}_RR$ is nonsingular with finite uniform dimension, the R is a left order in a finite direct product of full matrix rings over division rings.

Proof. Assume that R satisfies the hypothesis of the theorem. Then since R has finite uniform dimension it contains an essential finite direct sum of uniform left ideals, say $U_1 \oplus \cdots \oplus U_n$, which is faithful since R is semiprime. For any ideal A and left ideal $U_i' \subseteq U_i$, $AU_i' = 0$ implies $AU_i = 0$ since R is semiprime and U_i is uniform. This can be used to show that $\text{Ann}(U_i)$ is a prime ideal, so that $\cap_{j=1}^m P_j = 0$, where $\{P_j\}_{j=1}^m$ is the set of distinct elements of $\{\text{Ann}(U_i)\}_{i=1}^n$. If $P_j = \text{Ann}(U_i)$ and $U_k U_i \neq 0$, then for some $x \in U_i$ the homomorphism $[u \mapsto ux] : U_k \to U_i$ is nonzero, so since U_k is uniform and U_i is nonsingular, U_k is isomorphic to a submodule of U_i, and thus $P_j U_k = 0$. Therefore $P_j U_k \neq 0$ implies $U_k U_i = 0$ and $U_k \subseteq P_j$, so if A_j is the ideal generated by the left ideals of $\{U_i\}_{i=1}^n$ which are annihilated by P_j, then $A_j \cap P_j = 0$ and $A_j \oplus P_j$ is essential in R. Since A_j projects to an ideal in the prime ring R/P_j, it must be essential, so $\oplus_{j=1}^m A_j \subseteq R \subseteq \prod_{j=1}^m R/P_j$, which shows that R must be an essential R-submodule of $\prod_{j=1}^m R/P_j$. The second half of the proof will show that R/P_j is a left order in $Q_{\max}(R/P_j)$, which is a full ring of matrices over a division ring, and then it is not difficult to show that R is a left order in $\prod_{j=1}^m Q_{\max}(R/P_j)$. By Proposition 3.9, R satisfies the descending chain condition on left annihilators since it is a subring of a left Artinian ring. Thus $P_j = \text{Ann}(U_i)$ must be minimal among annihilators of finite subsets of U_i, and so there exists an exact sequence $0 \to R/P_j \to U_i^k$ for some positive integer k. This shows that R/P_j is nonsingular, and it can be assumed that $\dim(R/P_j) = k$.

This reduces the proof to the case in which R is prime, nonsingular, and has $\dim(R) = k$, with a uniform left ideal U for which there exists an exact sequence $0 \to R \to U^k$. If $\sigma = \text{rad}_{E(U)}$ and $V = U_\sigma$, then V is injective by Proposition 2.2, and so $\text{rad}_{E(R)} = \text{rad}_V$. Furthermore, V must then have dimension k over its endomorphism ring D, which is a division ring since U is monoform, and since V is finitely generated over D, $Q_{\max}(R) = \text{Bic}({}_R V)$ is the ring of $k \times k$ matrices over D^{opp}.

If $q \in Q_{\max}(R)$, then $B = Rq^{-1}$ is an essential left ideal of R, so let B_i be the intersection of B and the i^{th} component of U^k. Using arguments from the first half of the proof, there is an exact sequence $0 \to R \to B_1^k$ ($\dim(R) = k$), and B_1 is isomorphic to a submodule of B_i, for all i. Thus $R \subseteq B_1^k \subseteq \oplus_{i=1}^k B_i \subseteq B$, and so B contains a left regular element b of R. But b is left regular in the Artinian ring $Q_{\max}(R)$ since R is essential, and hence b is invertible there. If $bq = a$, then $q = b^{-1}a$, with $a, b \in R$, and so R is a left order in $Q_{\max}(R)$.

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