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An Introduction to Universal Localization at Prime Ideals

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In the study of commutative Noetherian rings, localization at a prime ideal has been a powerful tool. Let me just recall that if P is a prime ideal of a commutative Noetherian ring, then there exists a local ring R_P and homomorphism $\lambda : R \rightarrow R_P$ such that if $c \in R \setminus P$ then $\lambda(c)$ is invertible in R_P . An important property of the ring R_P is that $R_P/J(R_P) \cong Q(R/P)$, where $Q(R/P)$ is the field of quotients of the domain R/P .

In the noncommutative case, there have been several different approaches to localization. If P is a prime ideal of the Noetherian ring R , we would like to construct a ring R_P in which the elements of $\mathcal{C}(P)$ are invertible (The set $\mathcal{C}(P)$ consists of the elements of R that are not divisors of zero modulo P .) One approach is to attempt to construct a ring of fractions, but this is possible only when $\mathcal{C}(P)$ satisfies the Ore condition. When this fails, one can often invert a subset of $\mathcal{C}(P)$ and still obtain a reasonable ring. This approach has been studied rather extensively.

Another approach focuses on the category of modules, and attempts to construct an appropriate localization in the categorical sense. It requires that the localization of R/P should be “torsionfree” but other than that, as much of $R\text{-Mod}$ as possible should become “torsion”. This construction always exists, but in general does not lead to a satisfactory ring of quotients.

In the commutative case, the inverse images of the powers of the Jacobson radical of the localization R_P form the “symbolic powers” of the prime ideal P . Furthermore, the kernel of the mapping from R to R_P is equal to the intersection of the symbolic powers of P . It is possible to give a definition of symbolic powers $P^{(n)}$ in the noncommutative case, and in 1967 Alfred Goldie constructed a localization as a certain subring of the inverse limit of the rings $Q_{cl}(R/P^{(n)})$. We note that in his construction the first step is to factor out the intersection of the symbolic powers.

In 1973, Paul Cohn initiated another method of localization in the paper “Inversive localisation in Noetherian rings.” The paper constructs the universal localization R_Γ at a prime ideal P of a Noetherian ring by inverting the set Γ of matrices that are not divisors of zero modulo P .

By Goldie's theorem, for each prime ideal there is a simple Artinian classical quotient ring $Q_{cl}(R/P)$, and since its construction commutes with formation of matrix rings, that is, since $Q_{cl}(M_n(R/P)) \cong M_n(Q_{cl}(R/P))$, the set Γ consists of the matrices that become invertible over $Q_{cl}(R/P)$. The ring R_Γ is the ring universal with respect to the property that For the canonical homomorphism $\lambda : R \rightarrow R_\Gamma$, if C is a matrix in Γ , then $\lambda(C)$ is invertible of R_Γ . By definition, R_Γ is the ring universal with respect to this property.

Among other interesting consequences of the construction of R_Γ , Cohn proved that $R_\Gamma/J(R_\Gamma)$ is naturally isomorphic to $Q_{cl}(R/P)$. I was able to show that $R_\Gamma/(J(R_\Gamma))^n$ is an Artinian classical ring of quotients of $R/P^{(n)}$. It is also true that the kernel of the canonical mapping $\lambda : R \rightarrow R_\Gamma$ must always contain the intersection of the symbolic powers, though it may not be equal to this intersection.

As a ring theorist used to inverting elements, constructing the universal localization by adjoining enough generators and relations to invert all of the matrices regular modulo P seemed to give an object that wasn't likely to be very manageable. In fact, the "top" of R_Γ is nice, but it is very difficult to determine the kernel of the canonical mapping. The module theoretic localization and the universal localization seem to me to be at opposite ends of some spectrum. In fact, the two constructions coincide if and only if the left Ore condition holds, in which case both constructions give the Ore localization.

I discovered that it *is* possible to give a definition of the universal localization that makes no mention of matrices or matrix-inverting homomorphisms. I would like to give that definition, then I will give some examples and a few basic properties, and finally I will talk about a different construction of the universal localization that looks a bit more like the traditional construction of a ring of fractions via equivalence classes of ordered pairs of elements. (Even here, matrices regular modulo P appear on center stage.)

I was able to show that R_Γ is characterized as the ring universal with respect to this property. The universal property that I mentioned above can also be used to show that Goldie's localization is a homomorphic image of the universal localization. Basically, he first factors out the intersection of the symbolic powers, and although the kernel of $\lambda : R \rightarrow R_\Gamma$ always contains the intersection of the symbolic powers, it can in general be smaller.

To give a flavor of the different techniques, I would like to mention a result that was the first thing that I was able to prove about R_Γ .

If J is an idempotent ideal contained in P , then J is torsion relative to Γ and hence is contained in the kernel of the canonical map $\lambda : R \rightarrow R_\Gamma$.

Proof. If J is generated as a left ideal by x_1, x_2, \dots, x_n , then we have an expression $x_i = \sum_{j=1}^n a_{ij}x_j$, where $a_{ij} \in J$. This leads to a matrix equation $(I - A)x = 0$, in which I is the $n \times n$ identity matrix. Then $I - A$ is congruent to the identity matrix modulo J , so it is certainly regular modulo J , and thus each entry of x is torsion relative to Γ . Since the generators of J map to zero, we see that J is contained in the kernel of the canonical mapping. \square

Among other consequences, this allows the computation of the universal localization at a maximal ideal of an Artinian ring. In this case $R_\Gamma = R/P^n$, if $P^{n+1} = P^n$.

Some examples are long overdue.

Example 1

Let R be the ring $\begin{bmatrix} \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$ and let P be the prime ideal $\begin{bmatrix} p\mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$.

This is a very reasonable prime ideal in a reasonable ring, but P is an idempotent ideal, and so for the localization $R_{\Gamma(P)}$ we just get R/P .

Let A and B be commutative rings, let $X = {}_B X_A$ be a B - A -bimodule, and let R be the ring of formal matrices $\begin{bmatrix} A & 0 \\ X & B \end{bmatrix}$. If $S = \begin{bmatrix} I & 0 \\ X & J \end{bmatrix}$ is a semiprime Goldie ideal of R , then $R_{\Gamma(S)}$ is naturally isomorphic to the ring of formal matrices $\begin{bmatrix} A_I & 0 \\ Y & B_J \end{bmatrix}$, where $Y = B_J \otimes_B X \otimes_A A_J$.

Proof. The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \mid d \in C(J) \right\}$ is easily seen to be a left denominator set in R . The corresponding torsion ideal of R is

$$\begin{bmatrix} 0 & 0 \\ \text{rad}_{C(J)}(X) & \text{rad}_{C(J)}(B) \end{bmatrix},$$

and the corresponding ring of left fractions can be shown to be

$$T_1 = \begin{bmatrix} A & 0 \\ B_J \otimes_B X & B_J \end{bmatrix}.$$

The set $\left\{ \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \mid d \in C(J) \right\}$ is a right denominator set in T_1 , and the corresponding ring of right fractions can be shown to be the ring

$$T_2 = \begin{bmatrix} A_I & 0 \\ B_J \otimes_B X \otimes_A A_I & B_J \end{bmatrix}.$$

The Jacobson radical of T_2 is $\begin{bmatrix} A_I I & 0 \\ Y & J B_J \end{bmatrix}$, and so

$$T_2/J(T_2) \cong A_I/A_I I \oplus B_J/J B_J \cong Q_{cl}(R/I) \oplus Q_{cl}(B/J) \cong Q_{cl}(R/S).$$

Let $\theta : R \rightarrow T$ be any ring homomorphism such that $T/J(T)$ is naturally isomorphic to $Q_{cl}(R/S)$. Then θ is a $C(S)$ -inverting homomorphism, so in particular θ inverts elements of the form $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, with $c \in C(I)$ and $d \in C(J)$, respectively. Since T_1 is universal with respect to homomorphisms which invert $\left\{ \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \mid d \in C(J) \right\}$, there is a commutative diagram as follows.

$$\begin{array}{ccc} & \gamma' & \\ R & \rightarrow & T_1 \\ & \theta \searrow & \downarrow \theta' \\ & & T \end{array}$$

By the similar argument, θ' extends to T_2 , and thus there is a commutative diagram as follows.

$$\begin{array}{ccccc} & \gamma' & & \gamma'' & \\ R & \rightarrow & T_1 & \rightarrow & T_2 \\ & \theta \searrow & \downarrow \theta' & \swarrow \theta'' & \\ & & T & & \end{array}$$

This shows that T_2 is naturally isomorphic to $R_{\Gamma(S)}$. \square

Example 2

Let R be the ring of lower triangular 2×2 matrices over the ring of integers Z , and let S be the semiprime ideal $\begin{bmatrix} pZ & 0 \\ Z & qZ \end{bmatrix}$, where p, q are distinct prime numbers. If $Z_{(p)}$ denotes the localization of Z at pZ , then $Z_{(p)} \otimes_Z Z \otimes_Z Z_{(q)} \cong Q$, the field of rational numbers. By the above proposition, the universal localization $R_{\Gamma(S)}$ is isomorphic to the ring $\begin{bmatrix} Z_{(p)} & 0 \\ Q & Z_{(q)} \end{bmatrix}$. In this ring the two-sided ideal $\begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix}$ is not finitely generated on either the right or left. Thus $R_{\Gamma(S)}$ need not be left or right Noetherian, even though R is a Noetherian ring finitely generated (as a module) over its center.

The definition and some basic properties

Let R denote a left Noetherian ring (with identity), and let S be a semiprime ideal of R . Consider the following conditions for a ring T and ring homomorphism $\phi : R \rightarrow T$.

J_1 : The ring $T/J(T)$ is a semisimple Artinian ring.

J_2 : $S = \phi^{-1}(J(T))$

J_3 : The ring $T/J(T)$ is a classical ring of left quotients of R/S , under the induced embedding $\bar{\phi} : R/S \rightarrow T/J(T)$.

J_4 : If $\theta : R \rightarrow T'$ is a ring homomorphism such that conditions J_1, J_2 , and J_3 are satisfied, then there exists a unique ring homomorphism $\theta' : T \rightarrow T'$ such that $\theta = \theta' \phi$.

Since condition J_4 states that T is universal with respect to conditions J_1 through J_3 , a standard argument shows that if there exists a ring satisfying conditions J_1 through J_4 , then it must be unique. Before considering the existence of such a ring, we give the relevant definition.

Definition 1 *Let R be a left Noetherian ring, with semiprime ideal S . A ring satisfying the above conditions J_1 through J_4 is called the universal localization of R at S , and will be denoted by R_S , with canonical homomorphism $\lambda : R \rightarrow R_S$.*

For any ideal I of R , the set of elements $c \in R$ that are regular modulo I will be denoted by $C(I)$. We need to extend this definition relative to S , as follows. For any positive integer n , let $\Sigma_n(S)$ denote the set of all matrices C such that C belongs to the $n \times n$ matrix ring $M_n(R)$ and the image of C in $M_n(R/S)$ is a regular element. This will be abbreviated by saying that C is regular modulo S . Note that $C \in \Sigma_n(S)$ if and only if the image of C is invertible under the canonical mapping from $M_n(R)$ into the left classical quotient ring $Q_{cl}(M_n(R/S)) \cong M_n(Q_{cl}(R/S))$. The union over all $n > 0$ of $\Sigma_n(S)$ will be denoted by $\Sigma(S)$.

The universal localization $R_{\Sigma(S)}$ of R at $\Sigma(S)$ is defined as the universal $\Sigma(S)$ -inverting ring. It can be constructed as follows (see [7] and [6] for details). For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Sigma(S)$, take a set of n^2 symbols $[d_{ij}]$, and take a ring presentation of $R_{\Sigma(S)}$ consisting of all of the elements of R , as well as all of the elements d_{ij} as generators; as defining relations take all of the relations holding in R , together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Sigma(S)$.

Theorem 2 *Let R be a left Noetherian ring. For any semiprime ideal S of R , the universal localization R_S exists, and is unique up to isomorphism.*

Proof. The uniqueness follows immediately from the definition. If $\Sigma(S)$ is the set of all square matrices that are regular modulo S , then Theorem 4.1 of [7] shows that the universal $\Sigma(S)$ -inverting ring $R_{\Sigma(S)}$ satisfies properties J_1 through J_3 . If $\phi : R \rightarrow T$ is any ring that satisfies conditions J_1 through J_3 , then for any matrix $C \in \Sigma_n(S)$ it follows that $\phi(C)$ is invertible modulo $M_n(J(T)) = J(M_n(T))$, and hence $\phi(C)$ is invertible in $M_n(T)$. Since $R_{\Sigma(S)}$ is the universal $\Sigma(S)$ -inverting ring, it satisfies condition J_4 . \square

Proposition 3 *Let R be a left Noetherian ring. If S is a localizable semiprime ideal of R , then the universal localization R_S coincides with the Ore localization of R at S .*

Proof. If $C(S)$ satisfies the left Ore condition, it is well-known that the ring of left quotients of R with respect to the multiplicative set $C(S)$ satisfies conditions J_1 through J_3 . Since this ring of left quotients is universal with respect to inverting elements in $C(S)$, the argument used in the proof of the previous theorem can be repeated. \square

Proposition 4 *Let R be a left Noetherian ring, with semiprime ideal S .*

(a) *The canonical mapping $\lambda : R \rightarrow R_S$ is an epimorphism in the category of rings.*

(b) *The ring R_S is flat as a right module over R if and only if S is a left localizable ideal.*

Proof. Part (a) follows from the characterization of R_S as the universal $\Sigma(S)$ -inverting ring. Part (b) is Corollary 3.2 of [2]. \square

Theorem 5 *Let R be left Noetherian, let N be the prime radical of R , and let $K = \ker(\lambda)$, for the canonical homomorphism $\lambda : R \rightarrow R_S$.*

(a) *The kernel K is the intersection of all ideals $I \subseteq N$ such that $C(N) \subseteq C(I)$.*

(b) *The ring R/K is a left order in a left Artinian ring, and R_N is naturally isomorphic to $Q_d(R/K)$.*

Proof. Parts (a) and (b) are Proposition 1.3 and Theorem 1.4 of [3], respectively. \square

It is shown in Example 4 of [2] that the universal localization at a semiprime ideal of a left Noetherian ring need not be left Noetherian. In fact, the ring given as an example is a Noetherian ring finitely generated (as a module) over its center. On the other hand, it is possible to determine conditions under which the universal localization is left Artinian.

Corollary 6 *Let R be left Noetherian, let S be a semiprime ideal of R , and let $K = \ker(\lambda)$, for the canonical homomorphism $\lambda : R \rightarrow R_S$.*

(a) *The universal localization R_S is left Artinian if and only if $S^n \subseteq K$ for some $n > 0$.*

(b) *If P is a minimal prime ideal of R , then R_P is left Artinian.*

Proof. See Theorem 1.5 and Corollary 1.6 of [2]. \square

The symbolic powers of S will be defined as in the commutative situation, by extending S^n to $R_S \lambda(S^n) R_S$ and then contracting back to R .

Definition 7 Let R be a left Noetherian ring, with semiprime ideal S , and let $\lambda : R \rightarrow R_S$ be the canonical homomorphism.

The n^{th} symbolic power of S , denoted by $S^{(n)}$, is defined as

$$S^{(n)} = \lambda^{-1}(R_S \lambda(S^n) R_S).$$

Proposition 8 Let R be a left Noetherian ring, with semiprime ideal S , and let $\lambda : R \rightarrow R_S$ be the canonical homomorphism.

(a) $S^{(n)} = \lambda^{-1}(J(R_S)^n)$.

(b) $S^{(n)}$ is the intersection of all ideals I such that $S^n \subseteq I \subseteq S$ and $C(S) \subseteq C(I)$.

(c) $C(S)$ is a left Ore set modulo $S^{(n)}$.

Proof. Parts (a) and (b) follow from Proposition 2.2 of [3]. Since $S/S^{(n)}$ is the prime radical of $R/S^{(n)}$, part (c) follows from part (b) and Small's Theorem. \square

A number of additional results can be proved for the symbolic powers of S . For example, for all positive integers n, m we have $S^{(n)}S^{(m)} \subseteq S^{(n+m)}$. For commutative Noetherian rings it is a standard result that $\ker(\lambda) = \bigcap_{n=1}^{\infty} P^{(n)}$. This fails in the noncommutative setting, as shown by the following example. Let R be the ring of lower triangular 2×2 matrices with entries from the rational numbers, in which the first entry on the diagonal has odd denominator. If S is the Jacobson radical of R , then R/S is semisimple Artinian, and so $R_S = R$ and $S^{(n)} = S^n$ for all n . Thus $\ker(\lambda) = (0) \neq \bigcap_{n=1}^{\infty} S^{(n)}$. The following proposition gives some positive information along these lines.

Proposition 9 Let R be a left Noetherian ring, with semiprime ideal S , and let $\lambda : R \rightarrow R_S$ be the canonical homomorphism. Then $\ker(\lambda) \subseteq \bigcap_{n=1}^{\infty} S^{(n)}$.

Proof. This follows from the fact that the symbolic power $S^{(n)}$ is the kernel of the canonical homomorphism from R into $R_S/J(R_S)^n$, and this homomorphism satisfies properties J_1 through J_3 in the definition of R_S . \square

Given a prime ideal P of a two-sided Noetherian ring R , and any positive integer n , the left symbolic powers H_n of P are defined by Goldie [9] as follows: $H_1 = P$, and by induction, H_n is defined as the two-sided $C(P)$

closure of PH_{n-1} . Lemma 2.3 of [3] shows that $P^{(n)} = H_n$, for any positive integer n .

Assume that R is Noetherian and let P be a prime ideal of R . For each positive integer n , let Q_n be the Artinian classical ring of quotients of $R/P^{(n)}$. Then there is a canonical epimorphism $Q_{n+1} \rightarrow Q_n$, for $n = 1, 2, \dots$. Let \widehat{Q} be the inverse limit of the rings $\{Q_n\}_{n=1}^{\infty}$ under these epimorphisms, and let $\mu : R \rightarrow \widehat{Q}$ be the induced homomorphism. Goldie's localization Q of R at P is defined as the intersection of all subrings Q' of \widehat{Q} such that $Q'/J(Q')$ is simple Artinian, $\cap J(Q')^n = (0)$, and $\mu(P) \subseteq J(Q') \subseteq J(\widehat{Q})$. The proof of Theorem 1 of [10] shows that $Q/J(Q) \cong Q_{cl}(R/P)$, $\cap_{n=1}^{\infty} J(Q)^n = (0)$, and $P^{(n)} = \mu^{-1}(J(Q)^n)$.

Theorem 10 *Let P be a prime ideal of the Noetherian ring R . Then Goldie's localization of R at P is isomorphic to $R_P / \cap_{n=1}^{\infty} J(R_P)^n$.*

Proof. See Theorem 2.4 of [2]. \square

Construction of quotient modules

Here are the principal references I will use:

P. Malcolmson, Construction of universal matrix localizations, Springer Lecture Notes in Math. No. 951, 1982, pp. 117-131.

J. A. Beachy, On universal localization at semiprime Goldie ideals, *Ring Theory, Proceedings of the Biennial Ohio State-Denison Conference, May, 1992*, World Scientific, 1993, pp. 41-57.

I found Malcolmson's construction via triples to be a very interesting alternative to the standard construction via generators and relations, but again many of his arguments were pretty far away from things I was used to. I looked for a way to make the construction look more familiar.

The construction can be done for more general sets Σ (of matrices). Professor Cohn showed that elements of $\Sigma^{-1}R$ are actually entries in inverses of matrices, so each element of $\Sigma^{-1}R$ has the form $q = e_i C^{-1} e_j$ for some matrix $C \in \Sigma$. It is easier to work with triples of the form $aC^{-1}b$, and then we can just as well use as our model triples of the form $aC^{-1}x$, where a has entries in R and x has entries in a module.

Outline: We first define an equivalence relation under which the set of ordered triples define a commutative semigroup. We then define a congruence on the semigroup that gives us an abelian group, and allows definition of a scalar multiplication. One of the main computational tools is a condition that resembles the Ore condition.

The module ${}_R X$ will be fixed; X^n denotes the direct sum of n copies of X ; I_n denotes the $n \times n$ identity matrix; a column vector $x \in X^n$ will be denoted by x^t ; Σ will denote a set of square matrices over R such that

- (i) Σ contains all permutation matrices;
- (ii) if $C, D \in \Sigma$, then $\begin{bmatrix} C & A \\ 0 & D \end{bmatrix} \in \Sigma$ for any matrix A ;
- (iii) if $C, D \in \Sigma$ and CD is defined, then $CD \in \Sigma$.

The set of matrices inverted by any ring homomorphism satisfies these properties. In particular, if R is left Noetherian and P is a prime ideal of R , then the set Γ of all square matrices regular modulo P satisfies the above conditions.

The elements of a module of quotients, denoted by X_Σ , will be constructed as equivalence classes of ordered triples (a, C, x^t) , where $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$ (for any positive n). The ordered triples are modeled on the element $aC^{-1}x^t$, where C is invertible, as would be the case over R_Σ . Let $a \in R^n$, $C \in \Sigma_n$, $x \in X^n$, $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$. If C and D are invertible, then we have the following identities.

- (i) $[a \ b] \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = aC^{-1}x^t + bD^{-1}y^t$
- (ii) $aU(VCU)^{-1}Vx^t = aC^{-1}x^t$, provided U, V are invertible
- (iii) $aC^{-1}0 = 0 = 0C^{-1}x^t$

An addition of triples is based on the first of these identities. The second motivates the definition of the initial equivalence relation for triples. We say that $(a, C, x^t) \equiv (b, D, y^t)$ if there exist invertible matrices U, V in Σ such that $b = aU$, $D = VCU$ and $y^t = Vx^t$. It is easily checked that this defines an equivalence relation. (The proof of transitivity uses the fact that Σ is closed under products.) We note that $(a, C, x^t) \equiv (b, D, y^t)$ only if C and D have the same size. Equation (iii) provides the motivation for the definition of the subsemigroup that induces the final equivalence relation.

Definition 11 Let (a, C, x^t) , (b, D, y^t) be ordered triples with $a, b \in R^n$, $C, D \in \Sigma_n$ and $x, y \in X^n$, for some positive integer n .

If there exist invertible $n \times n$ matrices U, V in Σ such that

$$b = aU, \quad D = VCU, \quad \text{and} \quad y^t = Vx^t,$$

then we say that (a, C, x^t) is congruent to (b, D, y^t) via U, V , written

$$(a, C, x^t) \equiv (b, D, y^t) \quad \text{via} \quad U, V.$$

For $a \in R^n$, $C \in \Sigma_n$ and $x \in X^n$, the notation $(a : C : x^t)$ will be used for the equivalence class of the ordered triple (a, C, x^t) under the equivalence relation \equiv . The set of all such equivalence classes, for all positive integers n , will be denoted by $\Sigma^{-1}X$.

We can think of the equivalence relation as a sort of cancellation property:

$$(b, D, y^t) = (aU, VCU, Vx^t) \equiv (a, C, x^t).$$

This cancellation of invertible matrices will be extended in Proposition 16 to cancellation of any matrices in Σ .

Proposition 12 The sum of elements $(a : C : x^t)$, $(b : D : y^t) \in \Sigma^{-1}X$ defined by

$$(a : C : x^t) + (b : D : y^t) = \left([a \quad b] : \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

yields an associative, commutative binary operation on $\Sigma^{-1}X$.

Definition 13 *The subset of $\Sigma^{-1}X$ consisting of all equivalence classes of elements of the form*

$$(e_1, E_1, 0), \quad \text{or} \quad (0, E_2, e_2^t), \quad \text{or} \quad \left([e_1 \ 0], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2^t \end{bmatrix} \right)$$

for some $e_1 \in R^m$, $E_1 \in \Sigma_m$, $E_2 \in \Sigma_n$, and $e_2 \in X^n$ will be denoted by $\Sigma_0^{-1}X$.

We note that the first two forms could be thought of as “degenerate” versions of the third, so that really we can think of $\Sigma_0^{-1}X$ as consisting of elements of $\Sigma^{-1}X$ of the third type.

Proposition 14 *For elements $\bar{x}, \bar{y} \in \Sigma^{-1}X$, the relation \sim defined by*

$$\bar{x} \sim \bar{y} \quad \text{if there exist } \bar{z}_1, \bar{z}_2 \in \Sigma_0^{-1}X \text{ such that } \bar{x} + \bar{z}_1 = \bar{y} + \bar{z}_2$$

is a congruence on the semigroup $\Sigma^{-1}X$. The set $\Sigma^{-1}X/\sim$ of equivalence classes of this congruence is an abelian group.

If C, D are invertible matrices such that $DA = BC$ for matrices A, B , then $AC^{-1} = D^{-1}B$ and so $aAC^{-1}x^t = aD^{-1}Bx^t$. This motivates the following lemma for triples $(aA : C : x^t)$ and $(a : D : Bx^t)$ such that $DA = BC$, a situation reminiscent of the left Ore condition. This lemma will prove to be very useful computationally.

Lemma 15 *Let $a \in R^m$, $C \in \Sigma_n$, $x \in X^n$, and let A be any $m \times n$ matrix over R . If there exist an $m \times n$ matrix B and a matrix $D \in \Sigma_m$ such that $DA = BC$, then*

$$(aA : C : x^t) \sim (a : D : Bx^t).$$

Proof. If $a, C, D, A, B,$ and x are as stated, then we have

$$\begin{aligned}
& (aA : C : x^t) \\
& \sim (a : D : 0) + (aA : C : x^t) \\
& = \left([a \ aA] : \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \end{bmatrix} \right) \\
& \cong \left([a \ aA] \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} : \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} : \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ x^t \end{bmatrix} \right) \\
& = \left([a \ 0] : \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} Bx^t \\ x^t \end{bmatrix} \right) \\
& = (a : D : Bx^t) + (0 : C : x^t) \\
& \sim (a : D : Bx^t).
\end{aligned}$$

This completes the proof. \square

Example 3

If S is a semiprime ideal of R for which $\mathcal{C}(S)$ is a left denominator set, and $\Sigma = \Gamma(S)$, then for each $(a : C : x^t) \in \Sigma^{-1}X$ there exist elements $y \in X$ and $d \in \mathcal{C}(S)$ such that $(a : C : x^t) \sim (1 : d : y)$. To see this, let $\lambda : R \rightarrow R_S$ be the classical left localization of R at $\mathcal{C}(S)$. We can assume without loss of generality that λ is one-to-one. Let $(a : C : x^t) \in \Sigma^{-1}X$. Then $\lambda(C)$ is invertible over R_S , so it is possible to find a common denominator $d \in \mathcal{C}(S)$ for the entries of $\lambda(a)\lambda(C)^{-1}$. Thus we have elements $d \in \mathcal{C}(S)$ and $b \in R^n$ such that $da = bC$, and then it follows from Lemma 15 that $(1 \cdot a : C : x^t) \sim (1 : d : bx^t)$.

Proposition 16 *Let $a \in R^n$, $C \in \Sigma_n$, and $x \in X^n$.*

(a) *If $b \in R^n$ and $y \in X^n$, then*

$$(a : C : x^t) + (a : C : y^t) \sim (a : C : (x + y)^t)$$

and

$$(a : C : x^t) + (b : C : x^t) \sim (a + b : C : x^t).$$

(b) *For any matrices P, Q such that $PC, CQ \in \Sigma_n$,*

$$(a : C : x^t) \sim (a : PC : Px^t) \quad \text{and} \quad (a : C : x^t) \sim (aQ : CQ : x^t).$$

(c) *For any $b \in R^m$, $D \in \Sigma_m$, $y \in X^m$ and any matrices A, B of the appropriate size,*

$$(a : C : x^t) \sim \left([a \ b] : \begin{bmatrix} C & A \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ 0 \end{bmatrix} \right)$$

and

$$(a : C : x^t) \sim \left([0 \ a] : \begin{bmatrix} D & B \\ 0 & C \end{bmatrix} : \begin{bmatrix} y^t \\ x^t \end{bmatrix} \right).$$

Theorem 17 *If $x, y \in X$, then in $\Sigma^{-1}X$ we have $(1 : 1 : x) \sim (1 : 1 : y)$ if and only if $x = av^t$ and $y = bw^t$ for some $a, b \in R^n$, $v, w \in X^n$ and some $n > 0$ such that there exist $C, D, P, Q \in \Sigma_n$ satisfying $aD = bQ$, $Cv^t = Pw^t$ and $CD = PQ$.*

The mapping $\lambda : R \rightarrow R_\Sigma$ given by $\lambda(r) = (1 : 1 : r)$ is a ring homomorphism which inverts the matrices in Σ . Theorem 17 provides a connection with the description of the kernel of $\lambda : R \rightarrow R_\Sigma$ given in Corollary 11.9 of the second edition of *Free Rings and Their Relations*.

Corollary 18 *Let $\lambda : R \rightarrow R_\Sigma$ be the canonical homomorphism. Then for $r \in R$ we have $r \in \ker(\lambda) \iff$ there is a relation of the form*

$$\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with $A_{21}, A_{22}, B_{12}, B_{22} \in \Sigma$.

Proof. \implies If $r \in \ker \lambda$, then $(1 : 1 : r) \sim (1 : 1 : 0)$, and so it follows from part (a) of the theorem that for some $n > 0$ there exist $a, b, c, d \in R^n$ and $C, D, P, Q \in \Sigma_n$ such that $r = ab^t$, $0 = cd^t$, $aD = cQ$, $Cb^t = Pd^t$ and $CD = PQ$. Then

$$\begin{bmatrix} a & c \\ C & P \end{bmatrix} \begin{bmatrix} b^t & -D \\ -d^t & Q \end{bmatrix} = \begin{bmatrix} ab^t - cd^t & -aD + cQ \\ Cb^t - Pd^t & -CD + PQ \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}.$$

This proves the ‘‘only if’’ part of the corollary. \square

Definition 19 *The set of equivalence classes of $\Sigma^{-1}X / \sim$ will be denoted by X_Σ . The notation $[a : C : x^t]$ will be used for the class of $(a : C : x^t) \in \Sigma^{-1}X$.*

Let $a, r \in R^n$, $C \in \Sigma_n$, $b \in R^m$, $D \in \Sigma_m$, and $y \in X^m$. If C, D are invertible, then we have the following identity, which motivates the definition of a scalar multiplication.

$$\begin{aligned} [a \ 0] \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} &= [a \ 0] \begin{bmatrix} C^{-1} & C^{-1} r^t b D^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^t \end{bmatrix} \\ &= [aC^{-1} \ aC^{-1} r^t b D^{-1}] \begin{bmatrix} 0 \\ y^t \end{bmatrix} \\ &= aC^{-1} r^t \cdot bD^{-1} y^t \end{aligned}$$

Proposition 20 *The scalar product of elements $(a : C : r^t) \in \Sigma^{-1}R$ and $(b : D : y^t) \in \Sigma^{-1}X$ defined by*

$$(a : C : r^t) \cdot (b : D : y^t) = \left([a \ 0] : \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

yields a well-defined, associative operation.

Lemma 21 *The following conditions hold for scalar multiplication.*

(a) *If $(a : C : r^t) \in \Sigma^{-1}R$ and $(1 : 1 : x) \in \Sigma^{-1}X$, then*

$$(a : C : r^t)(1 : 1 : x) \sim (a : C : r^t x) .$$

(b) *If $(1 : 1 : s) \in \Sigma^{-1}R$ and $(a : C : x^t) \in \Sigma^{-1}X$, then*

$$(1 : 1 : s)(a : C : x^t) \sim (sa : C : x^t) .$$

Lemma 22 Let $\bar{p} = (u : P : r^t)$, $\bar{q} = (v : Q : s^t) \in \Sigma^{-1}R$, and let $\bar{x} = (a : C : x^t)$, $\bar{y} = (b : D : y^t) \in \Sigma^{-1}X$. Then

$$(\bar{p} + \bar{q})\bar{x} \sim \bar{p}\bar{x} + \bar{q}\bar{x} \quad \text{and} \quad \bar{q}(\bar{x} + \bar{y}) \sim \bar{q}\bar{x} + \bar{q}\bar{y}.$$

Theorem 23 The set R_Σ is an associative ring with identity, and for any module ${}_R X$, the set X_Σ is a unital left module over R_Σ .

We say that the module ${}_R X$ is Σ -torsionfree if $Cx^t = 0$ implies $x = 0$, for all $C \in \Sigma_n$ and all $x \in X^n$. We say that X is Σ -divisible if for each $x \in X^n$ and each $C \in \Sigma$ there exists $y \in X^n$ such that $Cy^t = x^t$.

Theorem 24 The homomorphism $\eta : X \rightarrow X_\Sigma$ is an isomorphism if and only if X is Σ -torsionfree and Σ -divisible.

Theorem 25 For any module ${}_R X$, the module of quotients X_Σ is naturally isomorphic to $R_\Sigma \otimes_R X$.

Proof of Lemma 22. We have the following equalities.

$$\begin{aligned}
(\bar{p} + \bar{q})\bar{x} &= ((u : P : r^t) + (v : Q : s^t))(a : C : x^t) \\
&= \left([u \ v] : \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} : \begin{bmatrix} r^t \\ s^t \end{bmatrix} \right) (a : C : x^t) \\
&= \left([u \ v \ 0] : \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right) \\
&= \left([u \ 0 \ v \ 0] \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
\bar{p}\bar{x} + \bar{q}\bar{x} &= (u : P : r^t)(a : C : x^t) + (v : Q : s^t)(a : C : x^t) \\
&= \left([u \ 0] : \begin{bmatrix} P & -r^t a \\ 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \end{bmatrix} \right) + \left([v \ 0] : \begin{bmatrix} Q & -s^t a \\ 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \end{bmatrix} \right) \\
&= \left([u \ 0 \ v \ 0] : \begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} : \begin{bmatrix} 0 \\ x^t \\ 0 \\ x^t \end{bmatrix} \right) \\
&= \left([u \ 0 \ v \ 0] : \begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} : \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \\ x^t \end{bmatrix} \right)
\end{aligned}$$

The two expressions are equal by Lemma 15, since

$$\begin{bmatrix} P & -r^t a & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & Q & -s^t a \\ 0 & 0 & 0 & C \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P & 0 & -r^t a \\ 0 & Q & -s^t a \\ 0 & 0 & C \end{bmatrix}.$$

Think: $DA = BC$ implies $(aA : C : x^t) \sim (a : D : Bx^t)$.

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