0.1 Some history

The study of commutative rings had its origin in three areas: number theory, algebraic geometry, and invariant theory. The ring $\mathbf{Z}[\mathrm{i}]$ was used by Gauss in a paper dating back to 1828, in which he proved that elements in the ring can be factored uniquely into products of "prime" elements. He then used this property to derive results on ordinary integers. For example, it is possible to use unique factorization in $\mathbf{Z}[\mathrm{i}]$ to show that every prime number congruent to 1 modulo 4 can be written as a sum of two squares. Even to prove results about the ring of integers it is useful to study larger sets of numbers, and so number theory itself had to be expanded to include new classes of rings.

In attempting to solve Fermat's last theorem, the rings $\mathbf{Z}[\zeta]$ (where ζ is a root of unity) were studied by Gauss, Dirichlet, and others. A flaw in this line of attack was soon discovered: unlike the situation in $\mathbf{Z}[\mathrm{i}]$, unique factorization may fail in the rings $\mathbf{Z}[\zeta]$. Dedekind introduced the notion of an ideal, and was able to show that even though individual elements might not have a unique factorization, each ideal can be expressed uniquely as a product of prime ideals. The theory of commutative rings, and the area called more generally "commutative algebra," have continued to contribute valuable techniques to number theory.

The introduction of Cartesian coordinates made it possible to connect geometry and algebra. After the introduction of complex numbers and the proof of the fundamental theorem of algebra, it became possible to clarify this connection. To any subset I of the polynomial ring $R = \mathbf{C}[x_1, \ldots, x_n]$ we can associate the *algebraic subset* of $\mathbf{C}^n$ consisting of zeros of I. That is,

$$Z(I) = \{(a_1, \ldots, a_n) \in \mathbf{C}^n \mid f(a_1, \ldots, a_n) = 0 \ \forall f \in I\} \ .$$

On the other hand, for any subset $X \subseteq \mathbf{C}^n$ we can define the subset

$$I(X) = \{f \in \mathbf{C}[x_1, \ldots, x_n] \mid f(x_1, \ldots, a_n) = 0 \ \forall (a_1, \ldots, a_n) \in X\}$$

of the ring R, which turns out to be an ideal (see Definition 1.1.7). This leads to the definition of the *coordinate ring* $A(X)$ of X, defined as $A(X) = R/I(X)$.

Hilbert's Nullstellensatz (1893) shows that the above constructions lead to a one-to-one correspondence between algebraic subsets of $\mathbf{C}^n$ and radical ideals of the polynomial ring R. This correspondence actually holds for any algebraically closed field. For a discussion of the implications of this correspondence in algebraic geometry, see Chapter 1 of Eisendub's book. Dedekind and Weber had shown in the early 1880's that when an algebraic set X is a curve the coordinate ring $A(X)$ has many properties in common with the rings of algebraic integers studied in number theory. This was probably the real beginning of the interaction between algebraic geometry and commutative algebra.

Motivated by the study of geometric properties of plane curves that remain invariant under certain classes of transformations, invariant theory became the study of the elements of the polynomial ring $R = F[x_1, x_2, \ldots, x_n]$ left fixed by the action of a group G of automorphisms of R. The fundamental problem was to find a finite system of generators for the subalgebra consisting of the set R^G of fixed elements. In a series of papers at the end of the 1800's, Hilbert solved the problem by giving an existence proof. The first step in the solution is now generally known as the Hilbert basis theorem (see Theorem 2.4.10).

Now let us turn to the noncommutative case. By stretching things a bit, the study of noncommutative rings can be dated back to Hamilton's discovery of the quaternions in 1843. At about the same time Grassmann began to study what are now called exterior algebras. It was realized by 1870 that the set of $n \times n$ matrices forms a ring. Noncommutative structures were studied under the name "hypercomplex systems." The formal definition of a ring came much later–it is usually attributed to Fraenkel, in a paper published in 1914.

The connection between hypercomplex systems and group representations was recognized by Molien in his work in the 1890's. In fact, Molien's thesis on the structure of hypercomplex systems studied what we would now call simple and semisimple algebras, and he showed that they could be decomposed into matrix algebras. In papers in 1897 he applied his results to group algebras over **C**, and proved a number of results in representation theory. In her 1929[1] paper, Emmy Noether recognized that a representation could be thought of as a module over the group algebra. She was able to develop the theory in greater generality, by working with rings satisfying the descending chain condition rather than just algebras over a field.

Emmy Noether's use of the ascending chain condition for commutative rings led to the study of noncommutative rings satisfying the same condition. It has turned out that this condition holds for important classes of noncommutative rings, in particular for certain group rings over infinite groups, and for associative rings that come up in studying Lie algebras. Goldie's theorem is now recognized as one of the major tools in studying noncommutative Noetherian rings, and Chapter 3 in the text (on the structure of noncommutative rings) ends with a proof of this theorem.

In making historical references to specific individuals, it seems appropriate to focus primarily on Emmy Noether, because of her influence on the direction of algebra, moving it toward the modern axiomatic presentation. It also seems appropriate to mention the department in which she spent most of her career, since her influence extended to the many visitors who came either as students or colleagues.

I'll begin the story with Gauss, who studied at the University of Göttingen from 1795 to 1798. In 1807, Gauss was appointed director of Göttingen's observatory and professor of mathematics, and he taught at the university until his death in 1855. Lejeune Dirichlet, who had held professorships at Breslau and Berlin, and was one of the most prominent mathematicians of the day, was chosen to succeed Gauss. It was very unfortunate that he survived Gauss by only four years. This time Riemann, who had stayed at Göttingen after getting his doctorate under Gauss in 1851, was chosen to succeed Dirichlet as holder of Gauss's chair. Riemann was a worthy successor, but his brilliant career was cut short by his death at age 39. Clebsch, who worked in algebraic geometry, taught at Göttingen from 1868 until his death in 1872.

The story continues with Richard Dedekind, who was the last student of Gauss at Göttingen, completing his doctorate in 1852. He became one of the leading algebraists of the latter part of the century, and although he was not a member of the department at Göttingen, he took a position in 1862 in his home town of Braunschweig, about fifty

[1] *Math. Zeitschr.* **30** (1929), 641–692

miles away. He was still at Göttingen when Dirichlet arrived, and he attended Dirichlet's lectures on number theory. He later edited his notes, and published them in 1863 as *Lectures on Number Theory*, under Dirichlet's name. In the third and fourth editions, in 1879 and 1894 respectively, he wrote supplements that gave an exposition of his own work on ideals. Dedekind also edited (with Weber) the collected works of Riemann, and he and Weber wrote a paper in 1882 which applied the theory of ideals to Riemann surfaces. I will say more about Dedekind's work in the introduction to Chapter One, which also included editing the collected works of Gauss. His fundamental work on algebraic number theory and his introduction of ideals influenced Emmy Noether very strongly.

Felix Klein accepted a position at Göttingen in 1886, determined to develop the department into a top research center. Under his leadership, the department began to eclipse the one in Berlin, and *Mathematische Annalen* (founded at Göttingen by Clebsch) gradually became more important than Crelle's *Journal*, whose editors were associated with the department in Berlin. In 1895, Klein persuaded his colleagues to hire David Hilbert away from Königsberg. Hilbert exerted an enormous influence on the department, as well as in the general mathematical community, and taught at Göttingen until 1930.

Emmy Noether was a student at Göttingen for one semester (during 1903/1904), although she received her doctorate at Erlangen, for work on invariant theory. In 1915 she was invited by Hilbert and Klein to return to Göttingen to work with Hilbert, as she was recognized as one of the leading experts on invariant theory. She remained there until forced by the Nazi government to leave in 1933, and exerted an influence much beyond her teaching role through the many visitors who came to Göttingen. These included Artin, for whom the class of Artinian rings is named, and Krull, who was responsible for much of the development of ideal theory in commutative rings. Richard Brauer, who made important contributions to representation theory, published two papers with Emmy Noether. Artin, Krull, and Brauer belonged to the next generation, who maintained the tradition begun by Noether at Göttingen. The introduction to Chapter Two contains more about her life and work.