

1.0 Introduction

Richard Dedekind was born in 1831 in Braunschweig, in what is now Germany. At age 16 he entered the Collegium Carolinum, where his father taught. It was at an intermediate level, between high school and university, and after two years he went on to the University of Göttingen. The mathematics department at that time focused on preparing secondary teachers rather than on research, and Dedekind evidently found the physics department more interesting. Gauss taught some courses in mathematics, but mostly at an elementary level.

In the fall of 1850, Dedekind attended his first course with Gauss, on least squares. He was very strongly influenced by Gauss's lectures, and went on to finish his doctoral work under his supervision. It only took him two years, but the short time reflected to a large extent the lack of broad training in current areas of mathematics.

Bernhard Riemann had entered Göttingen in 1846, but had spent two years studying in Berlin, before finishing his thesis in 1851, under Gauss. Both Dedekind and Riemann obtained their habilitation degrees in 1854, which entitled them to teach. While both were preparing the dissertation and lectures for the habilitation, Dedekind benefited greatly from Riemann's knowledge, and was able to fill in many of the gaps in his education. While he was lecturing at Göttingen, he studied papers written by Galois, and gave what may have been one of the first courses on Galois theory.

In 1855 Lejeune Dirichlet was appointed to fill the position left vacant by the death of Gauss. He came to Göttingen from Berlin, where he taught at both the University of Berlin and the Military College. (He had had some difficulties obtaining a teaching position because he had never formally obtained a doctorate. This was eventually solved by an honorary doctorate from the University of Cologne.) Dirichlet had maintained a personal as well as professional relationship with Gauss. He had also had a strong influence on Riemann, during the time that Riemann spent visiting Berlin. Unfortunately, the friendship of Dedekind, Dirichlet, and Riemann was ended by the untimely death of Dirichlet in 1858.

Dirichlet's first publication was on the case $n = 5$ of Fermat's last theorem. Although making important contributions to number of areas within mathematics, he kept his interest in number theory, and taught courses in number theory at Göttingen. Dedekind attended Dirichlet's lectures, and wrote up his notes after Dirichlet's death. The book *Vorlesungen über Zahlentheorie*¹ was published in 1863. Although it appeared under Dirichlet's name, and Dedekind himself always referred to it as Dirichlet's book, it was written entirely by Dedekind. Dedekind did attach his own name to several very famous supplements that he included with the 1879 and 1894 editions.

Dedekind obtained a position in Zurich in 1858, but returned to Braunschweig in 1862 to teach at his alma mater, which in the mean time had been upgraded to the Braunschweig Polytechnikum. He maintained contact with Göttingen, which was about fifty miles away, but he apparently found it very comfortable to live with his family (he never married) in his home town.

¹*Lectures on Number Theory*

One of Dedekind's significant contributions to mathematics was the work he did in editing the collected works of Gauss, Dirichlet, and Riemann. His own work was certainly influenced by this intimate knowledge of the work of acknowledged masters. In 1882 he published a joint paper with Weber which showed that certain coordinate rings in geometry have many properties in common with rings of algebraic integers. This application of ideal theory to the arithmetic theory of Riemann surfaces came out of their study of Riemann's work, and was the beginning of modern algebraic geometry.

My interest in Dedekind lies in his introduction of the notion of an ideal. He did this in supplements to the third and fourth editions of *Vorlesungen über Zahlentheorie*, where he worked in the context of what we would now call the ring of integers of an algebraic number field. His aim was to repair the breakdown caused by the lack of unique factorization in rings that had originally been thought to hold the key to a proof of Fermat's last theorem.

Example 1.4.1 of the text shows that the ring $\mathbf{Z}[\sqrt{-5}]$ is not a unique factorization domain. This ring is a featured example in Dedekind's book *Sur la Théorie des Nombres Entiers Algébriques*, which has been translated by J. Stillwell² I will try to summarize enough of Stillwell's introduction to whet your appetite for reading the entire translation, which still provides a fascinating introduction to some of the ideas of algebraic number theory. Most of the biographical sources note Dedekind's ability to formulate and express his ideas with exceptional clarity.

Fermat discovered about 1640 that an odd prime number can be represented as the sum of two squares iff it is congruent to 1 modulo 4. In 1654 he extended this to the results that if p is an odd prime, then $p = x^2 + 2y^2$ iff $p \equiv 1$ or $3 \pmod{8}$, and $p = x^2 + 3y^2$ iff $p \equiv 1 \pmod{3}$. The next step is to ask when $p = x^2 + 5y^2$, and in 1744 Euler conjectured that $p = x^2 + 5y^2$ iff $p \equiv 1$ or $9 \pmod{20}$. But sometimes $x^2 + 5y^2$ represents 2 times a prime, and Euler also conjectured that $p \equiv 3$ or $7 \pmod{20}$ iff $2p = x^2 + 5y^2$. The key to this bipolar disorder was found by Lagrange in 1773. He investigated which integers could be represented by a quadratic form $ax^2 + bxy + cy^2$. He found that it is possible to make substitution that doesn't change the set of integers that is represented, but changes the quadratic form to $ax^2 + b'xy + cy^2$, where $|b'| \leq a' \leq c'$. In this process the discriminant $b^2 - 4ac$ is left unchanged. This makes it possible to classify the forms that have negative discriminant: if it is -4 the form is equivalent to $x^2 + y^2$; if it is -8 the form is equivalent to $x^2 + 2y^2$; and if it is -12 the form is equivalent to $x^2 + 3y^2$. But corresponding to the discriminant -20 there are two reduced forms: $x^2 + 5y^2$, and $2x^2 + 2xy + 3y^2$. In the final analysis, this is responsible for the failure of unique factorization in the ring $\mathbf{Z}[\sqrt{-5}]$. Read Stillwell's presentation to get the whole story!

Dedekind is also connected with the beginning of group representations. The definition of a group character was first given in the late 1870's by Dedekind. He defined a *character* on a finite abelian group G to be a homomorphism from G to the multiplicative group \mathbf{C}^\times of nonzero complex numbers, and orthogonality relations were discovered at an early stage. Dedekind also defined what he called the "group determinant", and noticed that it can be factored nicely, when the group is abelian.

²*Theory of Algebraic Integers*, Cambridge University Press, 1996

In an 1896 letter to Georg Frobenius, Dedekind suggested that it might be possible to generalize this factorization in the nonabelian case. In 1897 Frobenius formulated the modern definition of a group character as the trace of a matrix representation (see Definition 4.1.1 and Definition 4.2.5). See T.Y. Lam's articles in the *Notices of the A.M.S.*³ for an interesting account of this interaction that led to the development of representation theory.

³1998, pp. 361–372 and 465–474