CHAPTER 1: RINGS
Review Problems

1. Let I be an ideal of the commutative ring R. Prove that I is a prime ideal iff I is the kernel of a ring homomorphism from R into a field.

   Solution: If $I = \ker(\phi)$, where $\phi : R \to F$ is a ring homomorphism, and F is a field, then the fundamental homomorphism theorem implies that R/I is isomorphic to a subring of the field F. It follows that R/I is an integral domain, and hence I is a prime ideal. (See Proposition 1.3.2 (b).)

   Conversely, if I is a prime ideal, then it is the kernel of the composite mapping defined by the projection $R \to R/I$ followed by the embedding of R/I into its field of quotients $Q(R/I)$. (See Theorem 1.3.5 for the construction of the field of quotients.)

2. Let R be a commutative ring that is not a field, and let P be a maximal ideal of R. Let $I = P[x]$ be the ideal of the polynomial ring $R[x]$ consisting of all polynomials in $R[x]$ with coefficients in P. Show that I is a prime ideal that is not maximal.

   Solution: Let F denote the factor ring R/P, which is a field. Define $\phi : R[x] \to F[x]$ by mapping x to x and mapping each coefficient a to $a + P$. This is a well-defined ring homomorphism (see Example 1.2.1), and it is clear that it is onto, with kernel I, so $R[x]/I$ is isomorphic to $F[x]$. This shows that I is a prime ideal, since $F[x]$ is an integral domain. On the other hand, x generates a proper nonzero ideal in $F[x]$, so the corresponding preimage in $R[x]$ is an ideal that properly contains I. Thus I is not a maximal ideal of $R[x]$.

   Note: This implies that $R[x]$ is not a principal ideal domain, since in a principal ideal domain every nonzero prime ideal is maximal.

3. Let D be a principal ideal domain, and let P be a prime ideal of D. Prove that D/P is a principal ideal domain.

   Solution: Since P is a prime ideal, the factor ring D/P is an integral domain. By Proposition 1.2.9, each ideal of D/P has the form I/P, for an ideal I with $P \subseteq I \subseteq D$. By assumption $I = aD$ for some $a \in D$, so it follows that I/P is a principal ideal since each coset $r + P$ has the form $ad + P$, for some $d \in D$.

4. Let R be a commutative ring with $1 \neq 0$. Prove that if every proper ideal of R is prime, then R is a field.

   Solution: We first note that R is an integral domain since the zero ideal is prime. Let a be a nonzero element of R. By assumption, the ideal a^2R is prime, and so $a^2 \in a^2R$ implies $a \in a^2R$. Thus $a = a^2r$ for some $r \in R$, and since R is an integral domain, we can cancel a to obtain $1 = ar$, showing that a is invertible.

5. Let F and K be fields. Prove that if $F[x] \cong K[x]$, then $F \cong K$.

   Solution: Let $\phi : F[x] \to K[x]$ be a ring isomorphism. First note that ϕ maps 0 to 0. The units of $F[x]$ are the nonzero elements of F. Since the corresponding statement holds in $K[x]$, and any isomorphism preserves units, it follows that ϕ maps F to K.

Thus ϕ restricts to a ring homomorphism from F into K. Similarly, the inverse ϕ^{-1} restricts to a ring homomorphism from K into F. It follows that the restriction of ϕ to F is the required isomorphism from F to K.

6. Let F be a field. Show that in the factor ring $F[x]/(x^n)$ an element $f(x)+(x^n)$ is invertible iff $f(0) \neq 0$.

   Solution: Note that the coset $x + (x^n)$ is a nilpotent element in the factor ring $F[x]/(x^n)$. Thus if $f(x) \in F[x]$ and $f(0) = 0$, then in the factor ring the element $f(x) + (x^n)$ is a sum of nilpotent elements. Exercise 1.4.1 of the text shows that in a commutative ring the set of nilpotent elements forms an ideal. If follows that if $f(0) = 0$, then $f(x) + (x^n)$ is a nilpotent element of $F[x]/(x^n)$, and therefore cannot be invertible.

   Conversely, if $f(0) \neq 0$, then $f(x) + (x^n)$ is the sum of an invertible element and a nilpotent element. Therefore $f(x) + (x^n)$ is invertible in $F[x]/(x^n)$. (Exercise 1.1.9 in the text states that if u is a unit and a is nilpotent, then $u - a$ is a unit; the same statment holds for $u + a$.)

7. Let $F = \mathbf{Z}_2$ be the field with two elements, and let R be the factor ring $F[x]/(x^2+1)$. Show that R has four elements, but that it is not isomorphic (as a ring) to either $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

   Solution: The ring R has four cosets, represented by 0, 1, x, and $x + 1$. Since R has characteristic 2, the underlying abelian group is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, so R cannot be isomorphic (as a ring) to $\mathbf{Z}_4$.

   Since F has characteristic 2, $x^2 + 1 = (x + 1)^2$. Thus $x + 1$ is a nilpotent element, while x is invertible, since $x^2 \equiv 1 \pmod{x^2 + 1}$. In $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ there are no nonzero nilpotent elements, and the only invertible element is $(1, 1)$, since $(1, 0)$ and $(0, 1)$ are zero divisors. Thus R cannot be isomorphic to $\mathbf{Z}_4$.

8. Let $F = \mathbf{Z}_2$ be the field with two elements. Show that the ring $R = F[x]/(x^3 + x)$ has exactly four proper nontrivial ideals.

   Solution: Over F, the polynomial $x^3 + x$ factors as $x(x + 1)^2$. The lattice of ideals of R corresponds to the lattice of ideals of $F[x]$ that contain the ideal $I = (x^3 + x)$. Since $F[x]$ is a principal ideal domain, this lattice corresponds to the lattice of factors of $x^3 + x$. Thus the proper nontrivial ideals of R are generated by the elements $x + 1$, $(x + 1)^2$, x, and $x(x + 1)$.

9. Show that if R is a division ring, then for any $a \in R$ the centralizer $C(a)$ is a division ring.

   Solution: An earlier exercise in the Section 1.1 of these class notes shows that the centralizer of an element a, defined by $C(a) = \{r \in R \mid ra = ar\}$, is a subring of R. We will repeat the proof. If $r, s \in C(a)$, then $ra = ar$ and $sa = as$. It follows that $(r+s)a = ra+sa = ar+as = a(r+a)$, $(rs)a = r(sa) = r(as) = (ra)s = (ar)s = a(rs)$, and $(-r)a = (-1 \cdot r)a = a(-1 \cdot r) = a(-r)$. Thus $C(x)$ is a subring of R.

If $y \in C(a)$ and $y \neq 0$, then $y^{-1}a = y^{-1}a(yy^{-1}) = y^{-1}(ay)y^{-1} = y^{-1}(ad)y^{-1} = (y^{-1}a)dy^{-1} = ay^{-1}$, and so $y^{-1} \in C(a)$, showing that $C(a)$ is a division ring.

10. Let D be an integral domain for which $IJ = I \cap J$ for all ideals I, J of D. Prove that D is a field.

    Solution: Let $a \in D$. Then $(aD)(aD) = aD \cap aD = aD$, so $a \in (aD)(aD)$, which implies that $a = \sum_{i=1}^{n}(ab_i)(ad_i)$ for some $b_i, d_i \in D$, for $1 \leq i \leq n$. Thus $a = a^2r$ for $r = (\sum_{i=1}^{n} b_i d_i)$. If $a \neq 0$, then we can cancel a to obtain $1 = ar$, showing that a is invertible.