

CHAPTER 3: STRUCTURE OF NONCOMMUTATIVE RINGS

Review Problems

1. Let R be a left Artinian ring in which $I^2 = (0)$ implies $I = (0)$, for all ideals I of R . Prove that R is a semisimple Artinian ring.

Solution: Let J be the Jacobson radical of R . Since R is left Artinian, we know that J is nilpotent. If $J \neq (0)$, let n be its index of nilpotence, so that by assumption we have $n \geq 2$. Then $(J^{n-1})^2 = (0)$, since $2n - 2 \geq n$, and this is a contradiction. We conclude that $J = (0)$, and hence R is a semisimple ring.

2. Let R be a semisimple Artinian ring. Prove that if $ab = 0$ implies $a = 0$ or $b = 0$, for all $a, b \in R$, then R is a division ring.

Solution: The quickest solution uses the Artin-Wedderburn theorem, which implies that R is a direct sum of matrix rings over division rings. If R is a proper direct sum (having more than one summand), say $R \cong S \oplus T$, then $(1, 0) \cdot (0, 1) = (0, 0)$, a contradiction. Thus $R = M_n(D)$, for some division ring D . If $n > 1$, we can easily find matrix units that contradict the assumption.

3. Let R be a ring, let ${}_R M$ be a Noetherian R -module, and let J be the Jacobson radical of R . Prove that either $J^n M = (0)$ for some positive integer n , or else the descending chain $M \supset JM \supset J^2 M \supset \cdots$ is an infinite chain, showing that M is not Artinian.

Solution: Suppose that $J^{n+1} M = J^n M$ for some positive integer n . Then since M is Noetherian, the submodule $J^n M$ is finitely generated, and so Nakayama's lemma implies that $J^n M = (0)$ since $J(J^n M) = J^n M$.

4. Show that if I and J are nilpotent ideals of the ring R , then the ideal $I + J$ is also nilpotent.

Solution: We claim that if n, m are positive integers with $I^n = (0)$ and $J^m = (0)$, then $(I + J)^{n+m-1} = (0)$. To show this, let $k = n + m - 1$, and suppose that $x_1 x_2 \cdots x_k$ is a product of elements $x_1, \dots, x_k \in I + J$. Then $x_i = a_i + b_i$, for some $a_i \in I$ and $b_i \in J$, so we must look at the product

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_k + b_k).$$

In the expansion of this product, each term has the form $y = y_1 y_2 \cdots y_k$, where either $y_i = a_i$ or $y_i = b_i$. If the number of terms a_i from I is t and $t \geq n$, then we can consider y to be an element of I^t , since I is an ideal and the terms b_j . Thus $y = 0$ since $I^t = (0)$. If $t < n$, then $t \leq n - 1$, so the number s of terms b_j from J must satisfy $s \geq m$, and then y is an element of $J^s = (0)$.

5. Let F be a field, let V be an infinite dimensional vector space over F , and let R be the idealization of V , as defined in Exercise 2.1.12 of the text. Compute the Jacobson radical of R .

Solution: We view R as the set of "formal" matrices of the form $\begin{bmatrix} a & 0 \\ v & a \end{bmatrix}$ such that $a \in F$ and $v \in V$. The ideal J consisting of all elements of the form $\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}$, such

that $v \in V$, satisfies $J^2 = (0)$, so it must be contained in the Jacobson radical of R . On the other hand, R/J is isomorphic to the field F , and so J is a maximal ideal. But J is a maximal left ideal since R is commutative, and so the Jacobson radical must be contained in J . We conclude that J is indeed the Jacobson radical of R .

6. Prove that any homomorphic image of a semisimple Artinian ring is again semisimple Artinian.

Solution: Assume that R is a semisimple Artinian ring. Then R is a sum of simple left R -modules, when viewed as a left module over itself. If $\phi : R \rightarrow S$ is any ring homomorphism, then $\phi(R) \cong R/\ker(\phi)$, so it suffices to consider factor rings R/I , where I is an ideal of R . The ring R/I is a left module over R , and the natural projection $\pi : R \rightarrow R/I$ is an R -module homomorphism as well as a ring homomorphism. It follows that the left R -module R/I must be semisimple as well. Finally, the module structures ${}_R(R/I)$ and ${}_{R/I}(R/I)$ are the same, so it follows that R/I is semisimple as a left module over itself. Therefore R/I is a semisimple Artinian ring.