## CHAPTER 2: MODULES
### Review Problems

1. Let $M$ be a left $R$-module. Show that $M$ is finitely generated if there exists a submodule $N \subseteq M$ such that $N$ and $M/N$ are both finitely generated.

2. Let $I, J$ be ideals of the ring $R$. Show that $R/I$ and $R/J$ are isomorphic as left $R$-modules if and only if $I = J$.

3. Show that if $x^2 = 0$ implies $x = 0$, for all $x$ in the ring $R$, then all idempotent elements of $R$ are central.

4. Let $S$ be a simple left $R$-module, and let $A$ be a minimal left ideal of $R$. Show that if $A \cdot S \neq (0)$, then $A$ and $S$ are isomorphic as left $R$-modules.

5. Let $R$ be a commutative ring with a unique maximal ideal $I$, and let $M$ be a nonzero finitely generated $R$-module. Show that $\mathrm{Hom}_R(M, R/I) \neq 0$.

6. Let $R$ be a ring, and let $M$ be a left $R$-module with submodules $N$ and $K$. Show that if $N$ and $K$ are Artinian, then so is $N + K$.

7. Compute the socle of the $\mathbf{Z}$-module $\mathbf{Z}_n$.

8. Let $R$ be a ring, and let $M$ be a left $R$-module that has a minimal submodule $S$ such that $M/S \cong S$. Prove that either $S$ is a direct summand of $M$, in which case $M \cong S \oplus S$, or else $S$ is the only proper nontrivial submodule of $M$.

9. Let $A$ and $B$ be finitely generated abelian groups. Prove that if $A \oplus A \cong B \oplus B$, then $A \cong B$.

10. Let $M$ be a finitely generated projective module over a principal ideal domain $D$. Prove that $M$ is a free $D$-module.

11. Let $R$ be a commutative ring, and let $M$ and $N$ be $R$-modules. Show that $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

12. Let $A$ be a nonzero injective $\mathbf{Z}$-module. Prove that $A$ cannot be finitely generated.