In preparation for the solved problems and supplementary exercises, we need to give some additional definitions.

DEFINITION. A sequence of $R$-homomorphisms

$$\cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is called an *exact sequence* if $\ker(f_i) = \text{Im}(f_{i-1})$, for each $i$.

The most basic examples of exact sequences come from submodules. If $N$ is a submodule of ${}_RM$, then we can form the factor module $M/N$, and we have two homomorphisms, the inclusion $i : N \to M$, and the projection $p : M \to M/N$. Since $\ker(p) = N = \text{Im}(i)$, we have the following exact sequence.

$$N \xrightarrow{i} M \xrightarrow{p} M/N$$

Since $i$ is one-eo-one, its kernel is zero, and since $p$ is onto, its image is $M/N$. This means that we can extend the sequence, by putting the zero module at the beginning and end, with the corresponding zero homomorphisms. This situation is important enough to deserve its own definition.

DEFINITION. An exact sequence of the form

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$$

is called a *short exact sequence.*

In the above exact sequence, the $R$-homomorphism $f_0$ must be one-to-one, while $f_1$ must be onto. Sometimes the sequences

$$0 \longrightarrow K \xrightarrow{f} M \qquad \text{and} \qquad M \xrightarrow{g} N \longrightarrow 0$$

are used to indicate that $f$ is one-to-one and $g$ is onto.

DEFINITION. A diagram of $R$-modules and $R$-homomorphisms, as shown below, is called a *commutative diagram* if $gf = qh$.

$$\begin{array}{ccc} K & \xrightarrow{h} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[q]{} & P \end{array}$$

**SOLVED PROBLEMS: SECTION 2.1**

13. Let $M$ be a left $R$-module, and let $M_1 \subseteq M_2 \subseteq \ldots \subseteq M$ be an ascending chain of submodules of $M$. Prove that $\bigcup_{i=1}^{\infty} M_i$ is a submodule of $M$.

14. Let ${}_RM$ be a left $R$-module, with submodules $N$ and $K$. Show that if $N \cup K$ is a submodule of $M$, then either $N \subseteq K$ or $K \subseteq N$.

15. Let $R$ be a commutative ring, and let $X$ be a subset of $R$ that contains 1 and is closed under products. Show that if $I$ is any ideal of $R$ with $I \cap X = \emptyset$, then there exists a prime ideal $P$ of $R$ with $I \subseteq P$ and $P \cap X = \emptyset$.

16. An module homomorphism $f : M \to N$ is called a *monomorphism* if it satisfies the following condition: if $g, h : X \to M$ are homomorphisms with $fg = fh$, then $g = h$. Prove that $f$ is a monomorphism if and only if it is one-to-one.

17. An module homomorphism $f : M \to N$ is called an *epimorphism* if it satisfies the following condition: if $g, h : N \to Y$ are homomorphisms with $gf = hf$, then $g = h$. Prove that $f$ is an epimorphism if and only if it is onto.

18. Show that if $p, q$ are distinct prime numbers, then there exists a short exact sequence

$$0 \longrightarrow \mathbf{Z}_p \xrightarrow{f} \mathbf{Z}_{pq} \xrightarrow{g} \mathbf{Z}_q \longrightarrow 0$$

of **Z**-modules.

19. In the following diagram, assume that the first square is a commutative diagram, and that both rows form exact sequences. Prove that there is a unique $R$-homomorphism $h_2 : M_2 \to N_2$ such that $h_2 f_1 = g_1 h_1$ (making the second square commutative).

$$\begin{array}{ccccccc} M_0 & \xrightarrow{f_0} & M_1 & \xrightarrow{f_1} & M_2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \vdots\,{\scriptstyle h_2} & & \\ N_0 & \xrightarrow[g_0]{} & N_1 & \xrightarrow[g_1]{} & N_2 & \longrightarrow & 0 \end{array}$$

20. Let $I$ be an ideal of the ring $R$ such that $I^n = (0)$, and let $M, N$ be left $R$-modules with an $R$-homomorphism $f : M \to N$.

    (a) Prove that $f$ induces an $R$-homomorphism $f' : M/IM \to N/IN$.

    (b) Prove that if $f'$ is onto, then $f$ is onto.