MATH 435

Spring 2009

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Review for

Exam # 2

1. Learn the derivations of the composite Trapezoidal and Simpson's rules and those of their error formulas.

2. Derive Simpson's rule by using

$$\int_{x_0}^{x_2} f(x)dx = a_0 f_0 + a_1 f_1 + a_2 f_2 + K f^{(4)}(n).$$

(Hint: Find a_0, a_1 and a_2 so that Simpson's rule is exact for all polynomials of degree less than or equal to 3 and then find K by applying the integration formula with $f(x) = x^4$.).

3. Determine the values of h and N to approximate the following integrals to within $\epsilon = 10^{-5}$ using both the composite trapezoidal and Simpson's rules:

(a) $\displaystyle\int_0^2 \frac{1}{x+4} dx$

(b) $\displaystyle\int_0^1 e^{-x^2} dx$

(c) $\displaystyle\int_0^2 e^{Rt} \sin 3x \, dx$

(d) $\displaystyle\int_0^\pi \sin x \, dx$

4. Apply Romberg integration to approximate $\int_0^1 x^{\frac{1}{3}} dx$ until $R_{n-1,n-1}$ and R_{nn} agree to within 10^{-4}.

5. Romberg integration is used to approximate

$$\int_0^1 \frac{x^2}{1+x^3} dx.$$

If $R_{11} = 0.25, \quad R_{22} = 0.2315$, find R_{21}.

6. Let

$$\begin{aligned} f(x) &= x \quad 0 \le x \le \frac{1}{2} \\ &= 1 - x \quad \frac{1}{2} \le x \le 1. \end{aligned}$$

Calculate approximations of $\int_0^1 f(x)dx$ using

(a) The trapezoidal rule over the interval $[0, 1]$

(b) The trapezoidal rule over the interval $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

(c) Simpson's rule over $[0, 1]$

7. Learn the derivation of Gaussian Quadrature formula with $n = 2$.

8. Determine the constants α, β, ν and δ so that the quadrature formula

$$\int_{-1}^{1} f(x)dx = \alpha f(-1) + \beta f(1) + \nu f'(-1) + \delta f'(1)$$

has degree of precision 3.

9. Approximate the following integrals using Gussian Quadrature with $n = 2$

   (a) $\int_0^{\frac{\pi}{4}} e^3 x \sin 2x dx$

   (b) $\int_3^{3.5} \frac{x}{\sqrt{x-9}} dx.$

10. Learn all the properties of Legendre and Chebyshev polynomials.

11. Suppose that $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ needs to be approximated by a polynomial $P_{n-1}(x)$ of degree $n-1$ such that

$$\max_{-1 \le x \le 1} |P_n(x) - P_{n-1}(x)|$$

is as small as possible. Then prove that

$$P_{n-1}(x) = P_n(x) - a_n \tilde{\tau}_n(x).$$

What is the minimum value?

Apply the above result to polynomial $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24}$ in $[-1, 1]$.

Find the minimum value.

12. What are the best possible choices of nodes for interpolation of $f(x) = e^x$ with a polynomial of degree at most 3 in $[2, 3]$?

13. Learn the Normal Equations Method for least-squares approximations of a set of discrete data and a function in an arbitrary interval.

14. Construct an interpolating polynomial of degree at most three to interpolate $f(x) = e^{-x^2}$ using zeros of an appropriate Chebyshev polynomial on $[2, 3]$. Repeat the problem with other functions, $f(x) = \sin 3x$, and $f(x) = \frac{1}{x}$.

15. Learn the process of power series economization with the zeros of Chebyshev polynomials. Practice the process with $f(x) = \sin x$ starting with a power series expansion of order 6 on $[-1, 1]$ and then reducing the degree while keeping error less than 0.01.

16. Learn the derivation of

    (a) Taylor's method of order n

    (b) Runge-Kutta method of order 2

    (c) Heun's method

17. Use Euler's method to approximate the solutions of

    (a) $y' = y - t^2 + 1, \ 0 \leq t \leq 2, \ y(0) = 0.5, \text{ with } h = 0.5$

    (b) $y' = \dfrac{y}{t} - \left(\dfrac{y}{t}\right)^2, \ 1 \leq t \leq 2, \ y(1) = 1 \text{ with } h = 01.$

    Compute both the local and global error bounds in each case.

18. Given the IVP: $y' = -2y$, $0 \leq t \leq 1$, $y(0) = 1$.

    Find an upper bound on the error at $t = 1$ in terms of the step size h. How small does h have to be to obtain an accuracy of $e = 10^{-5}$ at $t = 1$?

19. Show that the Midpoint method, the modified Euler's method and Heun's method applied to

$$y' = -y + t + 1, \quad 0 \leq t \leq 1, \ y(0) = 1$$

    give the same approximations for any choice of h.