MATH 435 Spring 2009

Review for Exam 1

1. Statement and Proof of Fixed-Point Iteration Theorem
2. Find Iteration function for each of these functions $f(x)$ to compute a zero in the specified interval and do two iterations in each case choosing a suitable x_0.
   (a) $f(x) = x - 2\sin x = 0, \quad [\frac{\pi}{3}, \frac{2\pi}{3}]$.
   (b) $f(x) = e^{-x} - \sin x = 0, \quad [.5, .7]$.
   (c) $f(x) = x^2 - x - 2, \quad [0, 7]$.
3. The function $f(x) = 1.5x - \tan x - 0.1$ has a root $\xi = 0.2059$. Calculate a sequence of converging $\{x_k\}$, choosing a suitable x_0.
4. (a) Derive Newton's algorithm using fixed-point iteration theorem.
   (b) Develop algorithms based on Newton's iteration, to compute $\sqrt[n]{A}$, and $\frac{1}{A}$. Apply your algorithms with some numerical values of A and suitable x_0.
   (c) Prove that Newton's method converges quadratically if x_0 is chosen close to the root.
   (d) Learn modified Newton's method for multiple roots. Do some simple examples.
   (e) Learn Aitken's acceleration scheme and use this scheme to the functions (b) and (c) of problem 2 to see how convergence is improved.
5. (a) Statement and Proof of Interpolation Error Theorem.
   (b) Determine the spacing h in a table of equally spaced values of $f(x) = \sqrt{x}$ between 1 and 2 so that interpolation with a second degree polynomial in the table will yield an accuracy of 5×10^{-8}.

   Hint:

   $$\max_{x\in[x_{i-1},x_{i+1}]} |(x - x_{i-1})(x - x_i)(x - x_{i+1})| = \frac{2h^3}{3\sqrt{3}}$$

   Answer: $\frac{-h^3}{24\sqrt{3}} < 5 \cdot 10^8 \Rightarrow h \approx 0.0128$
6. Find a bound for the error in linear interpolation.

   (**Answer:** $(x - x_0)^2 \frac{M}{8}$, where $|f''(x)| \leq M$ on $[x_0, x_1]$).

7. (a) Learn Newton's Interpolation formula using both Divided and Forward Differences.

   (b) Prove by **induction** on i:

$$f[x_0, \ldots, x_n] = \frac{1}{n!h^n}\Delta^n f_0$$

8. Apply both Newton's forward and backward difference formulas to interpolate $f(x) = e^x$ and $f(x) = \ln x$ in $[1, 2]$ with $n = 2$ and 3.

9. A clamped cubic spline polynomial for a function $f(x)$ is

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x \le 2, \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

   Given $f'(1) = f'(3)$. Find a, b, c and d.

10. A clamped cubic spline s for a function f is defined by

$$s(x) = \begin{cases} s_0(x) = 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \le x < 1, \\ s_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

   Find $f'(0)$ and $f'(2)$.

11. Prove formulas in 9(a) and 9(b) from Exercises of Chapter 2 of Lecture Notes.