Computational Methods for Feedback Control in Damped Gyroscopic Second-order Systems [1]

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Abstract

Two new computationally viable algorithms are proposed for partial pole-placement and eigenstructure assignment in damped gyroscopic matrix second-order systems.

The algorithms can be implemented with only a partial knowledge of the spectrum and the corresponding eigenvectors of the associated open-loop quadratic matrix pencil. Mathematical results are proved to guarantee that there will be no-spill over effects; that is, it is shown mathematically that the large-number of eigenvalues and eigenvectors that need to remain unchanged will not be effected by feedback. Furthermore, the algorithms work directly in second-order setting allowing them to exploit structures such as sparsity, bandness, symmetry etc. These practical features of the algorithms make them ideally suited for real-life applications such as control and stabilization of large flexible space structures.

1 Introduction

Vibrating structures such as beams, buildings, bridges, highways, large space structures, etc., are distributed parameter systems. While it is desirable to obtain a solution of a vibration problem in its own natural setting of distributed parameter systems; due to the lack of appropriate computational methods, in practice, very often a distributed parameter system is first discretized to a matrix second-order model using techniques of finite elements or finite differences and then an approximate solution is obtained from the solution of the problem in the discretized second-order model. A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

$$M\ddot{x}(t) + (D+G)\dot{x}(t) + Kx(t) = 0, \qquad (1)$$

where M, D, G, and K are respectively mass, damping, gyroscopic and stiffness matrices.

The system represented by (1) is called **damped gyroscopic system**. The gyroscopic matrix G is always skew-symmetric (that is, $G^T = -G$) and for all practical purposes, the mass matrix M can be assumed to be symmetric and positive definite ($M = M^T > 0$). In special cases where D and K are also symmetric and positive definite, the system is called **symmetric definite system**. If the gyroscopic force is not present, then the system is called **non-gyroscopic**.

The eigenvalues of the system (1) are the eigenvalues of the quadratic pencil:

$$P(\lambda) = \lambda^2 M + \lambda(D+G) + K.$$

To combat undesirable effects of vibrations such as resonances, caused by a few "bad" eigenvalues of the system, one needs to reassign those few "bad" eigenvalues, leaving the rest unchanged, by using a suitable feedback control force. This problem is known as the *partial pole placement problem* in control theory.

In certain situations when the system responses need to be altered by feedback, both eigenvalue assignment and eigenstructure assignment should be considered. This is because the eigenvalues determine the rate at which the system response decays or grows and the eigenvectors determine the shape of the response. The problem of assigning both eigenvalues and eigenvectors by feedback is called Eigenstructure Assignment Problem. When only a few undesirable eigenvalues and the associated eigenvectors of a second-order model need to be reassigned keeping the remaining eigenvalues and eigenvectors invariant by application of feedback, the problem is called *partial eigenstructure assignment problem* for the associated quadratic matrix pencil. Unfortunately, if the control matrix B is given a priori, the eigenstructure assignment may not be completely solvable [1].

Let a control force of the form

$$f = Bh(t)$$

where B is the input (control) matrix, be applied to the vibrating structure and the control vector $h(t)$ be

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chosen, using information on measured structure's position and velocity vectors, as

$$h(t) = F_1 \dot{x}(t) + F_2 x(t).$$

Then the closed-loop system corresponding to (1) is

$$M\ddot{x} + (D + G - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0. \quad (2)$$

The closed-loop matrix quadratic pencil corresponding to (2) is

$$P_c(\lambda) = \lambda^2 M + \lambda(D + G - BF_1) + (K - BF_2) \quad (3)$$

Mathematically, these problems are defined as follows. For notational convenience, we write $C = D + G$, throughout the rest of the paper.

Problem 1 (Partial Pole-placement Problem for Damped Gyroscopic Second-order Systems).
Given

1. *Real $n \times n$ matrices $M = M^T > 0$, C, K.*
2. *Real $n \times m$ $(m < n)$ control matrix B.*
3. *The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$, $p < 2n$ of the set of open-loop eigenvalues $\{\lambda_1, \dots, \lambda_{2n}\}$ of the pencil $P(\lambda)$ and the corresponding left eigenvector set $\{y_1, \dots, y_p\}$.*
4. *The self-conjugate set $\{\mu_1, \dots, \mu_p\}$ of scalars.*

Find

Real feedback matrices F_1 and F_2 of order $m \times n$ such that the spectrum of the closed-loop pencil (3) is the set $S = \{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$.

Problem 2 (Partial Eigenstructure Assignment Problem for Damped Gyroscopic Second-order Systems).
Given

1. *Real $n \times n$ matrices $M = M^T > 0$, C, K.*
2. *The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$, $p < 2n$ of the set of the open-loop eigenvalues $\{\lambda_1, \dots, \lambda_{2n}\}$ of the pencil $P(\lambda)$ and the corresponding left eigenvector set $\{y_1, \dots, y_p\}$.*
3. *The self-conjugate sets of scalars $\{\mu_1, \dots, \mu_p\}$ and the set of vectors $\{x_{c1}, \dots, x_{cp}\}$, such that $\mu_j = \overline{\mu_k}$ implies $x_{cj} = \overline{x_{ck}}$.*

Find

Real control matrix B of order $n \times m$ $(m < n)$ and real feedback matrices F_1 and F_2 of order $m \times n$ such that the spectrum of the closed-loop pencil (3) is the set $S = \{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$ with $\{x_{c1}, \dots, x_{cp}; x_{p+1}, \dots, x_{2n}\}$ as the associated eigenvector set, where $x_{p+1}, \dots, x_{2n}$ are the eigenvectors of $P(\lambda)$ corresponding to $\lambda_{p+1}, \dots, \lambda_{2n}$.

An obvious approach to solve the problems is to recast the problems in terms of first-order systems and then apply one of the many standard techniques available for first-order models. Unfortunately, there are some serious computational problems with this approach. These include the inversion of the mass matrix M, which may be ill-conditioned, and the loss of structures such as definiteness, sparsity, bandness, etc., which are often assets with large practical problems.

A second approach, which is popularly known as the *Independent Modal Space Control* (IMSC) approach [7, 8] also suffers from some practical computational difficulties. The most serious of them are the potential danger of spill-over and the stringent requirement that needs to be satisfied on actuators and sensors [7] for control decoupling.

In the last few years, a novel approach that circumvents all the above practical and computational difficulties, was proposed for solutions of both Problem 1 and Problem 2 (see [3-6]). *However, these papers dealt only with the special case where M, and K, and D are symmetric and positive definite. Furthermore, in none of these papers, gyroscopic forces were included in the model*, although such forces are playing crucial role in applications such as aircraft maneuvering. The major obstacle in including gyroscopic forces in those papers was that the orthogonality relations between the quadratic pencil (originally proposed in [3]), that formed the basis of solutions of these problems in the above-mentioned papers, could not be carried over to the damped gyroscopic case in a straightforward fashion.

A new orthogonality relation between the eigenvectors of a damped gyroscopic second-order model is obtained in this paper; and based on this relation, new algorithms for Problem 1 and Problem 2 are proposed for the second-order system (1) in the most general case. These new algorithms, like the previous ones in the special cases, are *"direct and partial modal"*. They are "direct" in the sense that the problems are solved directly in second-order setting; no transformation to the standard first-order state-space system is invoked. They are "partial-modal" in the sense that only a partial knowledge of eigenvalues and eigenvectors of the open-loop pencil is required for implementation of the algorithms. The "direct" nature allows the algorithms to take advantage of the special structure such as sparsity, symmetry, bandness, etc., and the "partial-modal" nature makes them very suitable for practical applications.

The orthogonality relations proved in [3, 4] as well as the previous algorithms in [3-6] for the symmetric definite non-gyroscopic quadratic pencil are recovered as special cases of the new orthogonality relation and the proposed algorithms in this paper.

2 Orthogonality Relations between the Eigenvectors of Quadratic Matrix Pencils

In this section, we first state the **new orthogonality relations** between the eigenvectors of a damped gyroscopic quadratic pencil. *The recent orthogonality results on the symmetric definite quadratic matrix pencil established in [3, 4] follow as special cases.* The result forms the basis of our proposed algorithms.

Theorem 3 (Orthogonality of the Eigenvectors of the Damped Gyroscopic Quadratic Pencil).
Let the eigenvalues of the pencil $P(\lambda) = \lambda^2 M + \lambda C + K$ *be partitioned into the disjoint sets* $\{\lambda_1, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$. *Let the right and left eigenvector matrices* X *and* Y, *and the eigenvalue matrix* Λ *of the pencil* $P(\lambda)$ *be partitioned conformably as* $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ *and* $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$ *with* $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $\Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$. *Then*

$$\Lambda_1 Y_1^H M X_2 \Lambda_2 - Y_1^H K X_2 = 0 \tag{4}$$

and

$$\Lambda_1 Y_1^H M X_2 + Y_1^H M X_2 \Lambda_2 + Y_1^H C X_2 = 0. \tag{5}$$

The following special case of Theorem 3 was obtained earlier [3] and can be recovered immediately from Theorem 3.

Corollary 4 (Orthogonality of the Eigenvectors of the Symmetric Definite Non-Gyroscopic Quadratic Pencil).
Consider the symmetric definite quadratic pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K, \tag{6}$$

where $M = M^T > 0$, $D = D^T$, $K = K^T$. *Let* Λ *be the eigenvalue matrix and* X *be the corresponding eigenvector matrix.*

Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ *are distinct, then the matrix*

$$\Lambda X^T M X \Lambda - X^T K X \tag{7}$$

is a diagonal matrix.

3 Computational Algorithms for Partial Eigenvalue and Eigenstructure Assignments in Damped Gyroscopic Second-order Systems

In this section, we first state results giving parametric solutions of Problem 1 and Problem 2 and, based on these results, two computational algorithms are proposed.

3.1 A Computational Algorithm for Problem 1

A parametric solution of Problem 1 is given by the following theorem.

Theorem 5 (Parametric Solution to the Partial Eigenvalue Assignment Problem in Damped Gyroscopic Systems).
Let the matrix B *have full rank. Let the scalars* $\mu_1, \ldots, \mu_p$ *and the eigenvalues* $\lambda_1, \ldots, \lambda_{2n}$ *of the open-loop quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *be such that the sets* $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, *and* $\{\mu_1, \ldots, \mu_p\}$ *are disjoint and each set is closed under complex conjugation. Let the pair* $(P(\lambda), B)$ *be partially controllable with respect to* $\{\lambda_1, \ldots, \lambda_p\}$. *Let* $\Gamma = (\gamma_1, \ldots, \gamma_p)$ *be a matrix such that*

$$\gamma_j = \overline{\gamma_k} \text{ whenever } \mu_j = \overline{\mu_k}\,. \tag{8}$$

Set $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$ *and* $\Lambda_{c1} = \text{diag}(\mu_1, \ldots, \mu_p)$. *Let* Z_1 *be the unique nonsingular solution of the Sylvester equation*

$$Z_1 \Lambda_{c1} - \Lambda_1 Z_1 = Y_1^H B\, \Gamma\,, \tag{9}$$

where Y_1 *denotes the matrix of left eigenvectors corresponding to* $\lambda_1, \ldots, \lambda_p$. *Then the real feedback matrices* F_1 *and* F_2 *given by*

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C)\,, \tag{10}$$

where Φ *satisfies the linear system*

$$\Phi Z_1 = \Gamma\,, \tag{11}$$

solve the partial eigenvalue assignment problem for the pair $(P(\lambda), B)$.

Conversely, if there exist real feedback matrices F_1 *and* F_2 *of the form (10) that solves the partial eigenvalue assignment problem for the pair* $(P(\lambda), B)$, *then the matrix* Φ *can be constructed satisfying (8) through (11).*

Remark 6 (Avoiding the Explicit Usage of the Damping Matrix in the Parametric Solution).
If, in addition to the conditions of Theorem 5, it is known that none of the eigenvalues $\lambda_1, \ldots, \lambda_p$ *is zero, then (10) can be simplified to*

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = -\Phi \Lambda_1^{-1} Y_1^H K\,. \tag{12}$$

Thus, in this case, explicit knowledge of the damping matrix is not needed in computing the feedback matrices. Since, as stated before, in practice it is hard to estimate damping, obtaining the feedback matrices without the explicit knowledge of the damping matrix is quite useful in practical applications.

Furthermore, note that the restriction that none of the eigenvalues $\lambda_1, \ldots, \lambda_p$ *is zero can be easily removed by a shifting procedure, as described in [3].*

Based on the Theorem 5, we now state the following algorithm:

Algorithm 7 (An Algorithm for Partial Pole Placement in Damped Gyroscopic Systems).

Step 1. *Form* $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $Y_1 = (y_1, \ldots, y_p)$, *and* $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$.

Step 2. *Choose arbitrary* $m \times 1$ *vectors* $\gamma_1, \ldots, \gamma_p$ *in such a way that* $\overline{\mu_j} = \mu_k$ *implies* $\overline{\gamma_j} = \gamma_k$ *and form* $\Gamma = (\gamma_1, \ldots, \gamma_p)$.

Step 3. *Find the unique solution* Z_1 *of the Sylvester equation*

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = Y_1^H B \Gamma\,.$$

If Z_1 *is ill-conditioned, then return to Step 2 and select different* $\gamma_1, \ldots, \gamma_p$.

Step 4. *Solve* $\Phi Z_1 = \Gamma$ *for* Φ.

Step 5. *If none of the* $\lambda_1, \ldots, \lambda_p$ *is zero, form*

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = -\Phi \Lambda_1^{-1} Y_1^H K\,,$$

otherwise form

$$F_1 = \Phi Y_1^H M \text{ and } F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H C)\,.$$

Remark 8 (Some Distinctive Feature of Algorithm 7).
The most distinctive feature of the algorithm is that it computes the solution of a large partial eigenvalue assignment problem by solving a small Sylvester equation and by using only the few eigenvalues of the associated large quadratic pencil that need to be reassigned and the corresponding right eigenvectors. Thus, the algorithm is readily applicable to control dangerous vibration in a structure, where only a small part of the spectrum needs to be reassigned and the rest is to remain unchanged. Furthermore, one can take complete advantage of the sparsity, symmetry, definiteness, etc., of the matrices M, C, *and* K *in computing* F_1 *and* F_2.

3.2 A Computational Algorithm for Eigenstructure Assignment in Damped Gyroscopic Systems

A parametric solution of Problem 2 is given by the following theorem. We omit the proof for lack of space. It will appear elsewhere.

Theorem 9 (Solution to the Partial Eigenstructure Assignment Problem in Damped Gyroscopic Systems).
Let the scalars $\mu_1, \ldots, \mu_p$ *and the eigenvalues* $\lambda_1, \ldots, \lambda_{2n}$ *of the open-loop quadratic pencil* $P(\lambda) = \lambda^2 M + \lambda C + K$ *be such that the sets* $\{\lambda_1, \ldots, \lambda_p\}$, $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$, *and* $\{\mu_1, \ldots, \mu_p\}$ *are disjoint and individually closed under complex conjugation. Let the "eigenvectors to be assigned"* $x_{c1}, \ldots, x_{cp}$ *and the "eigenvectors to be kept invariant"* $x_{p+1}, \ldots, x_{2n}$ *form a two-fold complete system (that is, the first-order realization of the closed-loop system would have a complete set of eigenvectors). Set* $X_{c1} = (x_{c_1}, \ldots, x_{c_p})$ *and* $\Lambda_{c1} = diag(\mu_1, \ldots, \mu_p)$. *Let* Y_1 *be as in Theorem 6.*

If the matrix

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1} \quad (13)$$

is nonsingular, then

(*i*) *The triplet of matrices* (B, F_1, F_2) *given by*

$$B = M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}, \quad (14)$$

$$F_1 = Z_1^{-1} Y_1^H M, \text{ and}$$
$$F_2 = Z_1^{-1}(\Lambda_1 Y_1^H M + Y_1^H C) \quad (15)$$

constitutes a (possibly complex) solution to the partial eigenstructure assignment problem.

(*ii*) *A solution with real* B_R, F_{1R}, *and* F_{2R} *is obtained from* B, F_1, *and* F_2 *as*

$$B_R = B T_c^H\,,\ F_{1R} = T_c F_1\,, \text{ and } F_{2R} = T_c F_2\,, \quad (16)$$

where the matrix T_c *is such that* $T_c^{-1} = T_c^H$ *and the matrices* $T_c \Lambda_{c1} T_c^H$ *and* $X_{c1} T^H$ *are real.*

Conversely, if there exists a triplet of real matrices (B, F_1, F_2) *that solves the eigenstructure assignment problem, then the matrix* Z_1 *defined by (13) is nonsingular.*

Remark 10 *Using Theorem 9, the solutions to partial eigenstructure assignment problem for the symmetric definite non-gyroscopic quadratic pencil proposed in [5] and [6] can be recovered.*

Algorithm 11 (An Algorithm for Partial Eigenstructure Assignment in Damped Gyroscopic Systems).

Step 1. *Form* $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $Y_1 = (y_1, \ldots, y_p)$, $\Lambda_{c1} = \mathrm{diag}(\mu_1, \ldots, \mu_p)$, *and* $X_{c1} = (x_{c1}, \ldots, x_{cp})$.

Step 2. *Form the matrix*

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1}.$$

Stop if Z_1 *is singular and conclude that the eigenstructure assignment with the given sets of eigenvalues and eigenvectors is not possible.*

Step 3. *Form the matrix* T_c *such that* $T_c \Lambda_{c1} T_c^H$ *is a real matrix.*

Step 4. *Form*

$$\begin{aligned} B &= (M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}) T_c^H, \\ F_1 &= T_c Z_1^{-1} Y_1^H M, \text{ and} \\ F_2 &= T_c Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C) \end{aligned}$$

by solving the appropriate linear systems.

Remark 12 *The most distinctive feature of the algorithm is that it computes the solution of a large partial eigenstructure assignment problem by solving a small linear algebraic system and by using only the few eigenvalues of the large quadratic pencil that need to be reassigned and the associated right eigenvectors.*

4 Numerical Experiments

In this section, we present results of our numerical experiments on some real-life data with Algorithms 7 and 11. Very satisfactory results have been obtained in both cases.

4.1 Vibrations of a Rotating Turbine Axle

Following [2, Example 20] we consider a large and sparse symmetric definite quadratic matrix pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ of order $n = 211$ modelling a rotating axle in a power plant, where masses are assumed to be symmetric with respect to the axle.

The matrix $M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$ is positive definite and the damping and stiffness matrices given by

$$D = (d_{ij}), \text{ where } d_{ij} = \begin{cases} -\gamma_i & , \quad i+1=j \\ \gamma_{i-1} + \delta_i + \gamma_i & , \quad i=j \\ -\gamma_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$K = (k_{ij}), \text{ where } k_{ij} = \begin{cases} -\kappa_i & , \quad i+1=j \\ \kappa_{i-1} + \kappa_i & , \quad i=j \\ -\kappa_j & , \quad i=j+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

with $\gamma_0 = \gamma_n = \kappa_0 = \kappa_n = 0$, are both symmetric tridiagonal.

Using the data provided in [2], the eigenvalues of the uncontrolled system are plotted in Figure 1. It is clear that the decay rate of the vibrations of the axle is governed by its most unstable eigenvalue

$$\lambda_1 = -1.3734 \cdot 10^{-6} \,,$$

which is marked in Figure 1 by "+" sign, whereas the other eigenvalues have much better stability properties, namely $\mathrm{Re}\,\lambda_j \leq -0.016267\,,\ j = 2, 3, \ldots, 422$.

4.2 Partial Eigenvalue Assignment for Rotating Turbine Axle

We choose a simple control matrix

$$B = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^T$$

and apply our Algorithm 7 to assign λ_1 to $\mu_1 = -0.016$ so that the decay rate is improved by the factor of 10^4.

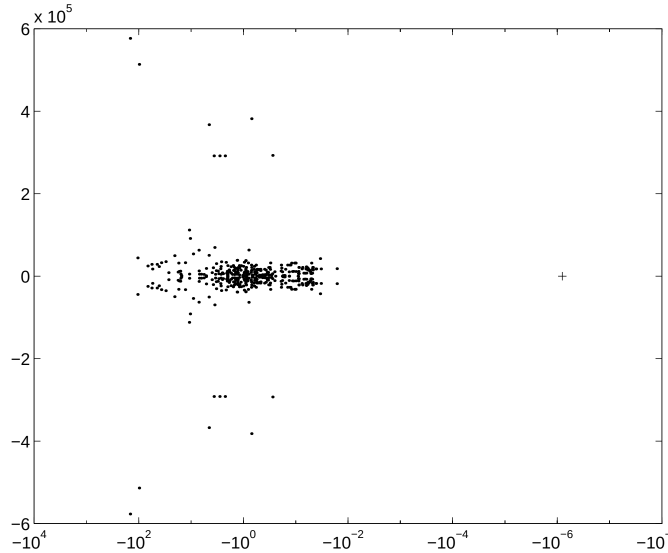

Figure 1: Open-Loop Eigenvalues of Rotating Turbine Axle; the Most Unstable Eigenvalue Is Marked by "+".

With random choice of the matrix $\Gamma = (-0.51454, -0.85747)^T$ the computed feedback matrices F_1 and F_2 are such that μ_1 was assigned correctly while the 2-norm of the difference between the other eigenvalues of the open-loop pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ and the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1) + (K - BF_2)$ is about $1.7 \cdot 10^{-6}$ (MATLAB was used to compute the eigenvalues). The 2×422 matrices F_1 and F_2 are not reproduced here because of the space limitation; however, we note that $||F_1||_2 < 116$ and $||F_2||_2 < 22$. Furthermore,

$$\frac{||F_1||_2}{||C||_2} < 0.57 \text{ and } \frac{||F_2||_2}{||K||_2} < 1.5 \cdot 10^{-11} \,,$$

which improves the result of [4] by reducing the control forces required to suppress the vibrations of the rotating turbine axle nearly 10^3-fold.

4.3 Partial Eigenstructure Assignment for Rotating Turbine Axle

Since the largest contribution to shape of the transient response of the vibrating system is generated by the eigenvector that corresponds to the most unstable eigenvalue of the system, we use Algorithm 11 to assign λ_1 to $\mu_1 = -0.016$ and, simultaneously, to assign the eigenvector corresponding to λ_1 to the vector

$$x_{c1} = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T \,,\ n = 211 \,.$$

Algorithm 11 produces the 211×1 control matrix B and with $||B||_2 < 2$ and the 1×211 feedback matrices F_1 and F_2 with $||F_1||_2 < 7.2$ and $||F_2||_2 < 1.4$, respectively, such that the prescribed eigenvalue and eigenvector are assigned correctly. Moreover,

$$\frac{||BF_1||_2}{||C||_2} < 0.07 \text{ and } \frac{||BF_2||_2}{||K||_2} < 1.8 \cdot 10^{-12}.$$

This shows that control forces required to suppress vibrations assigning the same eigenvalue are 10 times less than those required by eigenvalue assignment with a priori control matrix B. To achieve this, however, we need more sophisticated actuators than those needed to implement the simple control force used in eigenvalue assignment.

The computed matrices B, F_1, and F_2 are such that the 2-norm of the differences between the remaining eigenvalues of the open-loop pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ and the corresponding ones of the closed-loop pencil $P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1) + (K - BF_2)$ is about $2.2 \cdot 10^{-6}$ (MATLAB was used to compute the eigenvalues).

5 Summary and Conclusions

Feedback control is used to remove dangerous vibrations in structures ranging from automobiles to high rise buildings and large spacecrafts. The natural mathematical model of such structures are distributed parameter systems. However, due to lack of viable numerical methods to solve many vibration or vibration control problems directly in distributed parameter systems, a standard practice is to discretize the distributed parameter system to a matrix second-order model and then find an approximate solution of the problem from the solution in the discretized second-order model.

Unfortunately, even for this second-best alternative, standard existing approaches such as solution via reduction to a first-order form or the Independent Modal Space Control (IMSC) approach is either numerically dangerous or is not practically implementable.

In the last few years, the authors and their collaborators have devised numerically viable and practical algorithms for two important feedback control problems; namely, the partial pole placement and eigenstructure assignment problems in matrix second-order systems. These algorithms work directly in matrix second-order settings and can be implemented using limited resources such as a small number of eigenvalues and eigenvectors (frequencies and mode-shapes) which can either be measured in a vibration laboratory or be computed using the state-of-the-art computational techniques. These algorithms, thus, circumvent the engineering and computational difficulties of the standard existing methods. They are practical even for large structures; and, computationally, can take advantage of the exploitable structures very often offered by practical problems.

These algorithms, however, so far, were confined to solutions of feedback control problems in certain special cases of the second-order system; namely, to the symmetric definite non-gyroscopic systems.

The new feedback algorithms in this paper generalized the previous ones by the authors (and their collaborators) to the most general case, keeping the above-mentioned desired features. The algorithms for the symmetric definite non-gyroscopic systems can now be recovered as special case of these new algorithms.

The development of the algorithms is based on a new orthogonality relation between the eigenvectors of a damped gyroscopic matrix second-order pencil, which, besides its role in algorithm development, may be of independent interest and is a contribution in numerical linear algebra literature in its own right.

It is hoped that the results of this paper will set a new direction of research on vibration control.

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