An Optimization Approach for Minimum Norm and Robust Partial Quadratic Eigenvalue Assignment Problems for Vibrating Structures

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Abstract

The Partial Quadratic Eigenvalue Assignment Problem (PQEVAP) concerns the reassignment of a small number of undesirable eigenvalues of a quadratic matrix pencil, while leaving the remaining large number of eigenvalues and the corresponding eigenvectors unchanged. The problem arises in controlling undesirable resonance in vibrating structures and in stabilizing control systems. The solution of this problem requires computations of a pair of feedback matrices. For practical effectiveness, these feedback matrices must be computed in such a way that their norms and the condition number of the closed-loop eigenvector matrix are as small as possible. These considerations give rise to Minimum Norm Partial Quadratic Eigenvalue Assignment Problem (MNPQEVAP) and Robust Partial Quadratic Eigenvalue Assignment Problem (RPQEVAP), respectively. In this paper we propose new optimization based algorithms for solving these problems. The problems are solved directly in a second-order setting without resorting to a standard first-order formulation so as to avoid the inversion of a possibly ill-conditioned matrix and the loss of exploitable structures of the original model. The invariance of the large number of eigenvalues and eigenvectors by application of feedback is guaranteed by a proven mathematical result. Furthermore, the gradient formulas needed to solve the problems by existing optimization techniques are computed using only the few eigenvalues and eigenvectors of the associated quadratic pencil that can be computed using the state-of-the-art computational techniques. Above all, the proposed methods do not require the reduction of the model order or the order of the controller, even when the underlying finite element model has a very large degree of freedom. These attractive features, coupled with minimal computational requirements, such as solutions of small diagonal Sylvester equations and small algebraic linear systems, make the proposed algorithms ideally suited for application to large real-life structures. Numerical results show significant improvement in feedback norms and in the condition number of the closed-loop system. Also, the closed-loop eigenvalues have acceptable accuracy.

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1 Introduction

It is well-known that vibrating structures, such as bridges, high rise buildings, aircrafts, spacecrafts, etc.—, may be modeled by a second-order linear matrix ordinary differential equation of the form:

$$\mathbf{M}\ \ddot{\mathbf{q}}(\mathbf{t}) + \mathbf{D}\ \dot{\mathbf{q}}(\mathbf{t}) + \mathbf{K}\ \mathbf{q}(\mathbf{t}) = \mathbf{f}(\mathbf{t}), \tag{1}$$

where M, D, K are constant real $(n \times n)$ matrices, q(t) and f(t) are real n vectors. The matrix M is called the *mass matrix*, D is called the *damping matrix* and K is called the *stiffness matrix.* The vector f(t) represents an external force, t represents time and n is an integer, called the *degrees of freedom* (DOF) of the system. In many applications, the matrices M, K and D are symmetric; furthermore, M is positive definite and K is positive semi-definite. We make the same assumptions about M, K and D.

The dynamics of the structures modeled by Eq. (1), are governed by the eigenvalues and eigenvectors of the quadratic matrix polynomial $P(\lambda)$ (see Refs. [1–3]) where

$$P(\lambda) \equiv M\lambda^2 + D\lambda + K \tag{2}$$

The matrix polynomial $P(\lambda)$ is called the *open-loop pencil* and its eigenvalues are called the *open-loop eigenvalues.* If M is non-singular, $P(\lambda)$ has 2n eigenvalues which are the roots of the equation $det(P(\lambda)) = 0$.

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In some cases we find that a small number of open-loop eigenvalues are undesirable, because their presence has a damaging effect on the system. Undesirable resonance in vibrating structures is a case in point. It is well known that if a vibrating structure is acted on by external forces that are vibratory in nature whose frequencies are equal to or close to the natural frequencies of the vibrating structure then the vibrations get greatly amplified. This phenomenon is called resonance. It is widely believed that the collapses of the Tacoma Bridge in Washington, the Broughton Bridge in UK and the dangerous wobbling of the Millennium foot Bridge in London were caused by resonance. Now, the natural frequencies of a vibrating structure are closely related to the open loop eigenvalues.

Vibrations can be controlled either by using passive control devices such as shock absorbers or vibration dampers or by applying a feedback control force by means of active controllers. Passive control devices are widely used because of their simplicity, low cost and easiness of use. However they have some practical limitations. On the other hand, because of the advances in sensors and actuators, the use of active vibration control is becoming more and more popular. Indeed, many structures in several countries have been recently built using active controllers. It may be noted that the *Kyobashi Seiwa Building* built in 1989 in Tokyo was the first building to be constructed with an active control system.

To implement an active control strategy, feedback control gains must be computed in real time. So it is crucial that such gains are computed in an efficient and numerically robust way. Suppose that a control force of the form $Bu(t)$, (see Refs. [3,25]) is applied to the structure. Here B is a given real $n \times m$ matrix $(m \leq n)$ and $u(t)$ is a real m-vector given by

$$u(t) = F^T \dot{q}(t) + G^T q(t), \tag{3}$$

and F and G are unknown, constant, real, $(n \times m)$ matrices called the *feedback matrices.* We assume that q(t) and $\dot{\mathrm{q}}$(t) can be measured. From Eqs. (1) and (3), we obtain the closed-loop system:

$$M\ddot{q}(t) + (D - B\ F^T)\dot{q}(t) + (K - B\ G^T)q(t) = f(t). \tag{4}$$

The dynamics of this closed-loop system are determined by the eigenvalues and eigenvectors of the quadratic matrix polynomial:

$$P_c(\lambda) \equiv \lambda^2 M\ + \lambda(D - B\ F^T)\ + (K - B\ G^T)\ = 0, \tag{5}$$

The quadratic matrix polynomial $P_c(\lambda)$ is called the *closed-loop pencil* and its eigenvalues are called the *closed-loop eigenvalues.* Now, since the natural frequencies are closely related to the open loop eigenvalues, in order to prevent resonance, the feedback matrices F and G are determined such that the closed-loop spectrum is the one that we obtain from the open-loop spectrum by replacing the small number of resonant eigenvalues, by suitably chosen ones while keeping the remaining eigenvalues and associated eigenvectors unchanged. *The last property, known as the* ***no spill-over*** *property, guarantees that the large number of unassigned modes will not themselves become resonant or unstable.* The problem of finding F and G is referred to as the ***Partial Quadratic Eigenvalue Assignment Problem*** (PQEVAP).

The partial pole assignment problem was first introduced by Porter and Crosley (see Ref. [26]) for the first-order model. The problem was later studied by Datta and Sarkissian (see Ref. [9]), and Saad (see Ref. [27]), and Datta and Saad (see Ref. [28]), also for the first-order system. One way of solving the PQEVAP is to transform the quadratic control problem to a standard first-order state-space problem and then apply one of the of the excellent methods, now available in the literature (see Ref.[4]), for the standard complete pole-placement or partial pole-placement problem. However, there are several computational concerns with this approach. For instance, it would require inversion of the mass matrix, which may be ill-conditioned. Also, this transformation would destroy all the exploitable structures inherent in most practical problems, such as–symmetry, positive definiteness, sparsity, bandness, etc. Furthermore, since the existing eigenvalue assignment methods are designed for only small and dense problems, the order of the second-order finite element model, which is usually very large, must be reduced, and this process will invariably give rise to instability due to controllability and observability spill-overs. Moreover, the target eigenvalues and eigenvectors computed from a reduced-order model often very much differ from those of the original model. Similarly, the state-of-the-art independent Modal Space-Control (IMSC) approach (see Ref. [11]) for vibration control, also suffers from practical engineering and computational limitations. The basic idea behind IMSC is to decouple the problem into two independent problems, solve these independent problems individually, and then piece the solutions together to get the solution of the original problem. The open-loop decoupling requires the complete knowledge of the eigenvalues and eigenvectors of the open-loop pencil, whereas the state-of-the-art computational techniques for quadratic eigenvalue problem, such as the Jacobi-Davidson method (see Ref.[10]), are capable of computing only a small number of extremal eigenvalues and eigenvectors. The closed-loop decoupling, on the other hand, requires some stringent requirements on sensors and actuators (see Ref.[11]). In view of these considerations, for a numerically effective solution of the PQEVAP, it is imperative that *the problem be solved in quadratic setting using only the small number of eigenvalues and eigenvectors* that are computable or measurable in a vibration laboratory, and without any a priori model reduction. Furthermore, in the absence of computational techniques for computing the whole spectrum and the associated eigenvectors of a quadratic matrix pencil, the no spill-over property in a practical computational setting, *must be established with the help of a mathematical theory.*

In the multi-input case (i.e. when $m > 1$), the solution of the PQEVAP is not unique. For practical effectiveness, we can take advantage of this fact by determining F and G in such a way that not only $P_c(\lambda)$ has the desired spectrum but also the system has some additional desirable features. An important practical consideration in the design of a vibration control system is to ensure robustness, that is insensitivity of the closed-loop eigenvalues to small perturbations in data. To achieve this, the feedback matrices should be computed in such a way that their norms are as small as possible and the closed-loop eigenvector matrix has the minimum condition number (see Ref. [4]). The smaller feedback norms also lead to smaller control signals which in turn lead to lesser energy consumption and lesser noise amplification (see Ref. [5]). The problem of finding the feedback matrices such that the closed-loop pencil has the desired spectrum and the feedback norms are as small as possible is known as the ***Minimum Norm Partial Quadratic Eigenvalue Assignment Problem*** (MNPQEVAP) and the problem of finding the feedback matrices such that the closed-loop pencil has the desired spectrum and the conditioning of the closed-loop eigenvalues is as good as possible is called the ***Robust Partial Quadratic Eigenvalue Assignment Problem (RPQEVAP)***.

These are clearly optimization problems. To solve them in an optimization setting, a parameterized family of feedback matrices must be generated and then the parametric matrix has to be chosen appropriately. In this paper we use a quazi Newton optimization algorithm (see Ref. [12]). A gradient formula is necessary for this. Usually, computing gradient formulas is a straight forward task. But in this case an additional challenge is *to compute them in terms of the small number of computable eigenvalues and eigenvectors.*

A *direct and partial-modal approach* for solving the PQEVAP was first proposed in the single-input case by Datta, Elhay, and Ram (see Ref. [2]) and then generalized to the multi-input case by Datta and Sarkissian (see Ref. [9]) and by Datta, Elhay, Ram (see Ref. [21]) but without any reference to the Sylvester equation. These papers, however, did not consider the aspects of minimizing feedback norms and closed-loop conditioning of the PQEVAP. Such problems were considered, for the first-order control systems, by Keel, Fleming, and Bhattacharyya (see Ref. [7]) Cavin III, and Bhattacharyya (see Ref. [8]) and Varga (see Ref. [5]). The only paper that has been published so far that deals solely with the robustness issue for PQEVAP is by Qian and Xu (see Ref. [17]). This algorithm is not optimization-based. Also, optimization-based algorithms for minimum-norm and robust eigenvalue assignment in the cubic case (that is when the underlying eigenvalue problem is for cubic matrix polynomials, rather than quadratic), have been developed by Datta, Lin, and Wang (see Refs. [18, 19]). However, implementation of these algorithms require complete knowledge of the open-loop eigenvalues and eigenvectors.

The major contributions of this paper are as follows:

- A parameterized algorithm for generating a family of feedback matrices for the PQEVAP via Sylvester matrix equations has been developed. The no spill-over property is guaranteed by establishing mathematical results.

- Gradient formulas for the MNPQEVAP, RPQEVAP and also for the problem of simultaneously reducing the feedback norms and the closed-loop condition number, have been developed. Such gradient formulas are essential for the quasi-Newton optimization method (BFGS method) employed in this paper.

The parametric expressions for feedback matrices and the gradient formulas for the optimization problems were computed in terms of only the of the small number of eigenvalues of the open-loop pencil that were to be reassigned and the corresponding eigenvectors. Our Sylvester equation approach for parameterized solutions turn out to be crucial for the development of gradient formulas. Numerical experimental results show that the considerable reductions in both the feedback norms and the closed-loop condition numbers can be achieved by our algorithms. The accuracy of the closed-loop eigenvalues is acceptable as well. Our results on RPQEVAP show that these are quite comparable with those obtained by Qian and Xu. This paper contains a detailed discussion on results relating to the RPQEVAP. A preliminary version of this paper has appeared (see Ref.[15]). In this paper the results for MNPQEVAP have just been stated without proof, the proofs may be found in Refs. [14, 23].

To conclude this section, we make the following observations :

(i) Implementation of the control law in Eq. (3) requires the complete knowledge of the state and velocity vectors. However, in practice some of these may not be measurable. The non-measurable ones must be estimated by constructing suitable observers for a second-order control system. In a recent paper by Carvalho and Datta (see Ref. [29]), a numerically effective algorithm has been proposed for state and velocity vector estimation of a second-order control system via its transformation to a descriptor control system.

(ii) Our results apply only to finite-element models having a finite number of DOF. However, it might be possible to use our scheme for vibration control in real-life models such as a beam or a plate that have infinite number of DOF, in conjunction with some passive control device. It is also worth mentioning in this context that Datta and Sarkissian(see Ref.[30]) have developed a scheme for feedback control in distributed gyroscopic systems and applied it to partial pole assignment, corresponding to the lowest frequencies of small oscillations of a traveling string. Further research on developing such schemes for other distributed parameters systems is in order.

2 Notations and assumptions

The following notations are established :

$\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$

= the matrix of eigenvalues of the open-loop pencil $P(\lambda)$,

$\mathbf{\Lambda_1} = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$

= the matrix of eigenvalues to be reassigned,

$\mathbf{\Lambda_2} = diag(\lambda_{p+1}, \ldots, \lambda_{2n})$

= the matrix of eigenvalues to remain invariant,

$\mathbf{\Lambda_1'} = diag(\mu_1, \mu_2, \ldots, \mu_p)$

= the matrix of new eigenvalues to replace those in Λ_1,

$\mathbf{X} = (x_1, x_2, \ldots, x_{2n})$

= the matrix of right eigenvectors of the pencil $P(\lambda)$,

$\mathbf{X_1} = (x_1, x_2, \ldots, x_p)$

= the matrix of eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$

$\mathbf{X_2} = (x_{p+1}, \ldots, x_{2n})$

= the matrix of eigenvectors corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$,

$\|\mathbf{X}\|_{\mathbf{F}} =$ *Frobenius norm of X.*

Assumptions The following assumptions, which are quite reasonable in practice, are made throughout the whole paper.

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\} \cap \{\mu_1, \mu_2, \ldots, \mu_p\} = \phi, \tag{6a}$$

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\} = \phi, \tag{6b}$$

$$0 \notin \{\lambda_1, \lambda_2, \ldots, \lambda_p\}. \tag{6c}$$

3 Some eigenvalue-eigenvector properties of the quadratic matrix pencil

In this section we present some useful properties of the quadratic matrix pencils $P(\lambda)$ and $P_c(\lambda)$ which we use later in this paper.

Theorem 3.1

(i) A scaler $\lambda \in C$ is an eigenvalue of the quadratic pencil $P(\lambda) = M\lambda^2 + D\lambda + K$ with right eigenvector x iff λ is an eigenvalue of the matrix

$$A = \begin{pmatrix} O_{(n\times n)} & I_{(n\times n)} \\ \\ -M^{-1}K & -M^{-1}D \end{pmatrix},$$

with right eigenvector

$$\hat{x} = \begin{pmatrix} x \\ \\ \lambda x \end{pmatrix}.$$

(ii) The eigenvalues of the closed loop pencil

$P_c(\lambda) \equiv M\ \lambda^2 + (D - B\ F^T)\ \lambda + (K - B\ G^T)$

are the same as the eigenvalues of the matrix $\widehat{A} = A - \widehat{B}\ \widehat{F}$, where, $\widehat{F} = (-G^T\ \ - F^T)$,

and $\widehat{B} = \begin{pmatrix} O_{(n\times m)} \\ \\ M^{-1}B \end{pmatrix}$.

Also, the matrix of right eigenvectors of $\widehat{A}$ is the matrix $Y = \begin{pmatrix} Y_1 & X_2 \\ \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{pmatrix}$,

where, $Y_1 = [y_1, y_2, \ldots, y_p]$, and y_i is the right eigenvector of the pencil $P_c(\lambda)$ corresponding to the eigenvalue μ_i.

Proof: Since λ is an eigenvalue of the matrix pencil $P(\lambda)$ with right eigenvector x we have $(M\lambda^2 + D\lambda + K)x = 0$.

Thus

$$A\hat{x} = \begin{pmatrix} \lambda x \\ \\ -M^{-1}(K + \lambda D)x \end{pmatrix} = \lambda \begin{pmatrix} x \\ \\ \lambda x \end{pmatrix}. \tag{7}$$

This proves result (i).

By result (i) it follows that the eigenvalues of $P_c(\lambda)$ are the same as the eigenvalues of the matrix

$$\begin{pmatrix} O_{(n\times n)} & I_{(n\times n)} \\ \\ -M^{-1}(K - BG^T) & -M^{-1}(D - BF^T) \end{pmatrix} = A - \widehat{B}\ \widehat{F}, \tag{8}$$

and the matrix of right eigenvectors of the matrix $A - \widehat{B}\ \widehat{F}$ is the matrix $Y = \begin{pmatrix} Y_1 & X_2 \\ \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{pmatrix}$.

Orthogonality properties of the eigenvectors of a quadratic matrix pencil

Generalizing the well-known orthogonality properties of the eigenvectors of a symmetric matrix and of a symmetric definite linear pencil of the form $(K-\lambda M)$ (see Ref. [20]), three orthogonality relations for the quadratic matrix pencil were derived by Datta, Elhay and Ram (see Ref. [2]). The relations were further modified by Datta and Sarkissian (see Ref. [1]). One of these relations, which is stated below, is used in this paper.

Theorem 3.2

Let $M = M^T > 0,\ K = K^T \geq 0$ and suppose Λ_1 and Λ_2 have disjoint spectra. Then

$$\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1^T K X_2 = 0. \tag{9}$$

Proof: See Refs. [2,23].

The above orthogonality relation can be rewritten as:

Corollary 1

$$\begin{pmatrix} -KX_1 \\ \\ MX_1\Lambda_1 \end{pmatrix}^T \cdot \begin{pmatrix} X_2 \\ \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)}. \tag{10}$$

This corollary can be used to prove the following result:

Corollary 2

If a $(2n \times p)$ matrix $\mathbb{Q}$ satisfies $\mathbb{Q}\begin{pmatrix} X_2 \\ \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)}$, then there exists a

$(p \times p)$ matrix Ψ such that

$$\mathbb{Q}^T = \begin{pmatrix} -KX_1 \\ \\ MX_1\Lambda_1 \end{pmatrix} .\Psi. \tag{11}$$

Proof: See Ref. [23]

4 A parametric expression for the feedback matrices

As a first step towards solving the RPQEVAP and the MNPQEVAP in an optimization setting, we proceed to obtain a parametric solution of the of the PQEVAP in terms of a parametric matrix Γ. Then in the following sections we describe how to choose Γ appropriately to solve the MNPQEVAP and RPQEVAP. One of the best-known numerical optimization schemes, the Broyden-Fletcher-Goldfarb-Shannon (BFGS) is used to solve the optimization problems. For more on the BFGS method, see Refs. [12, 13].

Theorem 4.1 (Parametric expression for feedback matrices)

Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \in \mathbb{C}^{m \times p}$ be such that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the $(p \times p)$ Sylvester equation:

$$\Lambda_1 Z - Z\Lambda_1' = -\Lambda_1 X_1^T B\Gamma. \tag{12}$$

Then feedback matrices F and G that solve the PQEVAP are given by

$$F = MX_1\Lambda_1\Phi^T \text{and } G = -KX_1\Phi^T, \tag{13}$$

where Φ is obtained by solving the linear system:

$$\Phi Z = \Gamma. \tag{14}$$

Moreover, for every non-zero Γ satisfying the above relation, the matrices F and G are real. *Proof.* The proof comes in four parts.

Part I **Proof of the no-spill-over property**

Here we prove that if the matrices F and G are chosen as above and Φ is any non-zero $(m \times p)$ matrix, then the eigenvalues of the closed-loop pencil would include $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}$ with corresponding eigenvectors $x_{p+1}, x_{p+2}, \ldots, x_{2n}$. That is, *with such choices of F and G, there will be no spill-over.*

Proof: We need to show that:

$$MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 = 0. \tag{15}$$

Eq. (15) can be rewritten as :

$$MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2 - B[F^TX_2\Lambda_2 + G^TX_2] = 0. \tag{16}$$

Now, since the matrices (Λ_1, X_1) and (Λ_2, X_2) are partial eigenvalue and corresponding eigenvector matrices, we have:

$$MX_1\Lambda_1^2 + DX_1\Lambda_1 + KX_1 = 0, \tag{17}$$

$$MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2 = 0. \tag{18}$$

So, by virtue of Eq. (18), we need only show:

$$B[F^T X_2\Lambda_2 + G^T X_2] = 0. \tag{19}$$

It is easy to see that this relation follows immediately by substituting the expressions of the feedback matrices F and G and then using the orthogonality relation in Theorem 3.1.

Part II **Partial assignment of the spectrum**

Here we show that if Φ is chosen according to the criterion stated in the theorem then the spectrum of the closed-loop pencil will include $\mu_1, \mu_2, \ldots, \mu_p$ and the eigenvectors corresponding to these eigenvalues will be $y_1, y_2, \ldots, y_p$.

Proof :

Let $Y_1 = [y_1, y_2, \ldots, y_p]$, where y_j is the eigenvector associated with μ_j.

In order to show that the closed-loop pencil has the eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ with $y_1, y_2, \ldots, y_p$ as the corresponding eigenvectors, we need to show that

$$MY_1\Lambda_1'^2 + (D - BF^T)Y_1\Lambda_1' + (K - BG^T)Y_1 = 0. \tag{20}$$

This means that we have to show that:

$$\begin{aligned}
&MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1,\\
&= B[F^TY_1\Lambda_1' + G^TY_1],\\
&= B\Phi[\Lambda_1X_1^TMY_1\Lambda_1' - X_1^TKY_1],\\
&= B\Phi Z,
\end{aligned} \tag{21}$$

where,

$$Z = \Lambda_1X_1^TMY_1\Lambda_1' - X_1^TKY_1. \tag{22}$$

Suppose we choose Φ such that $\Phi\ Z = \Gamma$, where,

$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ is chosen arbitrarily.

Then, we must show that:

$$MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1 = B\Gamma, \tag{23a}$$

$$or,\ (M\mu_j^2 + D\mu_j + K)y_j = B\gamma_j \quad for\ (j = 1 : p). \tag{23b}$$

By assumption (6a) it follows that μ_j (for $j = 1 : p$) is not an eigenvalue of $P(\lambda)$.

$$\text{Hence, } \det(M\mu_j^2 + D\mu_j + K) \neq 0. \tag{24}$$

This means that y_j's are uniquely determined by the Eq. (23b). Thus for our choice of F, G, Y_1 and Φ Eq. (20), is satisfied.

Part III **Solution of the Sylvester equation** (12) **is unique**

We now prove that Z is the unique solution of the Sylvester equation

$\Lambda_1Z - Z\Lambda_1' = -\Lambda_1X_1^TB\Gamma.$

Pre-multiplying both sides of Eq. (23b) by $-\Lambda_1 X_1^T$ we get

$$-\Lambda_1 X_1^T M Y_1 \Lambda_1'^2 - \Lambda_1 X_1^T D Y_1 \Lambda_1' - \Lambda_1 X_1^T K Y_1 = -\Lambda_1 X_1^T B\Gamma. \tag{25}$$

Now eliminating D from this equation using Eq. (17) we obtain

$-\Lambda_1 X_1^T B\Gamma = \Lambda_1 Z - Z\Lambda_1'$

Also, by virtue of the assumption (6a) the diagonal matrices Λ_1 and Λ_1' have no eigenvalues in common, thus the Sylvester Eq. (12) has a unique solution.

Part IV **Feedback matrices are real**

Finally, we show that if Γ is chosen such that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$ then, F and G are real.

Since $\{\mu_1, \mu_2, \ldots, \mu_p\}$ is a self-conjugate set, $\bar{\Lambda}_1'$ can be obtained from Λ_1' by interchanging certain pairs of columns and the corresponding rows.

Thus, there exists a permutation matrix P_1 such that

$$\bar{\Lambda}'_1 = P_1^T \Lambda_1' P_1. \tag{26}$$

Again, by virtue of the condition that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$ it follows that $\bar{\Gamma}$ can be obtained from Γ by interchanging the same pairs of columns as was done to obtain $\bar{\Lambda}_1'$ from Λ_1. Thus,

$$\bar{\Gamma} = \Gamma P_1. \tag{27}$$

Also, since $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is also a self-conjugate set, there exists a permutation matrix P_2 such that

$$\bar{\Lambda}_1 = P_2^T \Lambda_1 P_2. \tag{28}$$

We know that if A is a real matrix then the eigenvectors of A, associated with complex-conjugate eigenvalues are themselves complex-conjugate.

Thus,

$$\bar{Y}_1 = Y_1 P_1 \; and \; \bar{X}_1 = X_1 P_2. \tag{29}$$

From above it follows

$$\bar{Z} = \bar{\Lambda}_1 (\bar{X}_1)^T M \bar{Y}_1 \bar{\Lambda}_1' - (\bar{X}_1)^T K \bar{Y}_1 = P_2^T Z P_1 \tag{30}$$

and

$$\bar{\Phi} = \Phi P_2. \tag{31}$$

These relations imply that $\bar{F} = F$ and $\bar{G} = G$.

It may be noted that although there does not exist a numerically verifiable necessary and sufficient condition for the non singularity of the solution Z of Eq. (12), (as also observed in Ref. [6]), for most non null choices of the matrix Γ, the matrix Z is non singular.

5 Minimizing the feedback norms

Let $\mathbb{I} = \frac{1}{2} \left\| S \right\|_F^2 \; = \frac{1}{2} \left[\left\| F \right\|_F^2 + \quad \left\| G \right\|_F^2 \right]$ where $S = [G^T \; F^T]$. Since F and G are functions of Γ only, the problem of minimizing $\mathbb{I}$ can be posed as:

Minimize: $\mathbb{I} = f(\Gamma)$.

This is an unconstrained optimization problem. To solve this problem using the Broyden-Fletcher-Goldfarb-Shannon (BFGS) algorithm we require an analytic expression for the gradient of $\mathbb{I}$ with respect to Γ. We denote this gradient by $\bigtriangledown_{\Gamma}(\mathbb{I})$. *Such a gradient formula is obtained in terms of only the known quantities*, namely, $\Lambda_1, \Lambda_1', X_1$, and B.

Below we just state the result without proof. The proof can be found in Refs. [14] and [15]. For easy reference, we first state a result on the solution of a Sylvester equation (see Ref. [22]) that was used to derive our gradient expression for MNPQEVAP and will also be used to obtain the gradient formula for RPQEVAP later in this paper.

Theorem 5.1

Suppose the Sylvester equation

$$AX - XB = C \tag{32}$$

has a unique solution. Let $\alpha(t)$ and $\beta(t)$ be coprime monic polynomials having degrees μ and ν respectively such that $\alpha(A)C = 0$ and $C\beta(B) = 0$. Then a unique solution of the Sylvester equation can be represented in the form :

$$X = \sum_{j=1}^{\nu}\sum_{i=1}^{\mu} \gamma_{ij} A^{i-1} C B^{j-1}, \tag{33}$$

where $(\gamma_{ij})'s$ are certain scalars of no significance to us.

Corollary 5.1 Suppose $m = n$ and the matrices A and B have n distinct eigenvalues, with no eigenvalues in common. Then for an arbitrary $n \times n$

matrix C, a unique solution of the Sylvester equation

$$BY - YA = C \tag{34}$$

is given by

$$Y = \sum_{j=1}^{n} \sum_{i=1}^{n} \gamma_{ij} B^{j-1}(-C)A^{i-1}. \tag{35}$$

Theorem 5.2 (Gradient formula for $\mathbb{I}$)

Suppose F and G are defined as in the previous theorem. Let $S = [\ G^T \quad F^T]$, $P = MX_1\Lambda_1$, $Q = -KX_1$, and $C = [\ Q^T \quad P^T\ ]$. Let Z satisfy the Sylvester equation:

$$\Lambda_1\ Z \ -\ Z\ \Lambda_1' = -\ \Lambda_1 X_1^T B\Gamma.$$

Suppose that Z is invertible and U satisfies the Sylvester equation:

$$\Lambda_1'\ U -\ U\ \Lambda_1 =\ Z^{-1}CS^H\Phi, \tag{36}$$

where $\Phi Z = \Gamma$. Then

$$(i)\ S = \Gamma Z^{-1}C, \tag{37a}$$

$$(ii)\ \ \bigtriangledown_{\Gamma}\ (\mathbb{I}) = \frac{1}{2}[\ Z^{-1}CS^H - U\Lambda_1 X_1^T B\ ]^T. \tag{37b}$$

Proof: See Refs. [14, 15, 23]

6 A gradient-based method for robust partial quadratic eigenvalue assignment

By the Bauer-Fike theorem (see Ref. [20]), an overall measure of the conditioning of the eigenvalues of the closed-loop matrix is provided by the condition number of the matrix Y. Now, the conditioning of the eigenvalues of this matrix is best when Y is unitary or orthogonal since in this case the condition number of Y (with respect to 2-norm) is 1. Thus we seek to determine $\mathbf{\Gamma}$ such that $J = \left\|(I - Y^H Y)^2\right\|_F^2$ is as small as possible. This measure of robustness was used before by Keel, Fleming, Bhattacharyya, (see Ref. [7]) for the first-order model. It has worked well in the first-order case and also in the quadratic case, as shown by results of our numerical examples here.

If Y were a unitary or orthogonal matrix then

$$\begin{pmatrix} Y_1 \\ \\ Y_1\Lambda_1' \end{pmatrix}^H \cdot \begin{pmatrix} X_2 \\ \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)}, \tag{38}$$

or, $Y_1^H X_2 + \bar{\Lambda}_1' Y_1^H X_2 \Lambda_2 = O_{(p\times 2n-p).}$

We now show that J can be split into two parts J_1 and J_2.

From Eq. (38) and Corollary 2 we obtain:

$$\begin{pmatrix} Y_1 \\ \\ Y_1\Lambda'_1 \end{pmatrix} = \begin{pmatrix} -K\bar{X}_1 \\ \\ M\bar{X}_1\Lambda_1 \end{pmatrix} .\Psi_1 = \begin{pmatrix} -K\bar{X}_1\Psi_1 \\ \\ M\bar{X}_1\Lambda_1\Psi_1 \end{pmatrix}. \tag{39}$$

Hence,

$$Y_1 = -K\bar{X}_1\Psi_1 = \bar{Q}\Psi_1, \tag{40}$$

and

$$Y_1\Lambda'_1 = M\bar{X}_1\Lambda_1\Psi_1 = \bar{P}\Psi_1, \tag{41}$$

where $P = MX_1\Lambda_1$ and $Q = -KX_1$.

Let $W_1 = I_p - Y_1^H Y_1 - \Lambda'_1 Y_1^H Y_1 \Lambda_1$

and $W_2 = I_{2n-p} - X_2^H X_2 - \bar{\Lambda}_2 X_2^H X_2 \Lambda_2$. Then by Eq. (38) we obtain:

$$(I - Y^H Y) = \begin{pmatrix} W_1 & O_{(p\times 2n-p)} \\ \\ O_{(2n-p\times p)} & W_2 \end{pmatrix}. \tag{42}$$

Therefore,

$$(I - Y^H Y)^2 = \begin{pmatrix} (W_1)^2 & O_{(p\times 2n-p)} \\ \\ O_{(2n-p\times p)} & (W_2)^2 \end{pmatrix}. \tag{43}$$

Then

$$J = \left\|(I - Y^H Y)^2\right\|_F^2 = \left\|(W_1)^2\right\|_F^2 + \left\|(W_2)^2\right\|_F^2 = J_1 + J_2 \ (say). \tag{44}$$

It may be noted, Eq. (38) relates $Y_1,\ X_2,\ \Lambda_2,$ and Λ_1'. Now, $Y_1 = [y_1, y_2, \ldots, y_p]$ is determined by the choice of $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ since $y_j's$ satisfy the equation $(M\mu_j^2 + D\mu_j + K)y_j = B\gamma_j$ for $(j = 1 : p)$. Γ is chosen by the user, whereas $(M\mu_j^2 + D\mu_j + K)$ and B involve quantities that are given in the problem. The matrix Λ_1' is also given. Since one of the objectives of the problem is to work without any knowledge of the matrices X_2 and Λ_2, Eq. (38) cannot be directly utilized in choosing Γ. It is possible that for the optimal Γ Eq. (38) will not be exactly satisfied. If $Y_1^H X_2 + \bar{\Lambda}_1' Y_1^H X_2 \Lambda_2 = W_3$, then the exact expression for J is

$$J = \left\|W_4\right\|_F^2 + \left\|W_5\right\|_F^2 + \left\|W_6\right\|_F^2 + \left\|W_7\right\|_F^2. \tag{45}$$

where $W_4 = (W_1)^2 + W_3 W_3^H$, $W_5 = (W_2)^2 + W_3^H W_3$, $W_6 = W_1 W_3 + W_3 W_2$, $W_7 = W_3^H W_1 + W_2 W_3^H$. In this equation W_3 and W_2 are unknown. By assuming Eq. (38) to be true we obtain Eq. (44), which is a simple approximation of this equation and this permits us to obtain the necessary gradients of J in terms of only those quantities that are known.

Now, the matrix X_2 is independent of Γ; $\Lambda_1',\ \Lambda_2$ are fixed matrices; and the matrix Y_1 is a function of the parameter Γ. Thus, J_1 is a function of Γ and J_2 is independent of Γ.

So, $\bigtriangledown_\Gamma(J_2) = 0$.

Hence, by Eq. (44)

$$\bigtriangledown_\Gamma(J) = \bigtriangledown_\Gamma(J_1). \tag{46}$$

Also, since J_2 remains invariant, therefore, J is as small as possible when J_1 is as small as possible. Thus in order to determine $\mathbf{\Gamma}$ for which J is as small as possible, we determine the $\mathbf{\Gamma}$ for which $\mathbf{J_1}$ is as small as possible using the BFGS method. For this we obtain the gradient of J_1 with respect to Γ.

Theorem 6.1 (Matrix gradient formula for J_1):

Let $Z_1 \equiv I_p - Y_1^H Y_1 - \bar{\Lambda}_1 Y_1^H Y_1 \bar{\Lambda}_1'$,

$Z_2 \equiv I_p - Y_1^H Y_1 - \Lambda_1' Y_1^H Y_1 \Lambda_1$,

$Z_3 \equiv Z_1^2 Z_2 + Z_2^2 Z_1 + \Lambda_1 Z_1^2 Z_2 \Lambda_1' + \bar{\Lambda}_1' Z_2^2 Z_1 \bar{\Lambda}_1$,

Let U_1 satisfy the Sylvester equation

$$\Lambda_1' \; U_1 - \; U_1 \; \Lambda_1 = -Z_3 Y_1^H K \bar{X}_1 C_1^{-1}. \tag{47}$$

where, $C_1 = P^T \bar{P} + Q^T \bar{Q}$, $P = MX_1\Lambda_1$ and $Q = -KX_1$.

Then,

$$\bigtriangledown_{\mathbf{\Gamma}}(\boldsymbol{J_1}) = \; \mathbf{2[U_1 \Lambda_1 X_1^T B]^T}. \tag{48}$$

From the definition of $J_1 = \left\| (W_1)^2 \right\|_F^2$, it follows that:

$J_1 = tr[\{(W_1)^2\}^H (W_1)^2]$

$= tr[(W_1^H)^2 (W_1)^2]$.

Thus $J_1 = tr[Z_1^2 \; Z_2^2]$.

So, $\Delta J_1 = tr[\; (\Delta Z_1^2) Z_2^2 + Z_1^2 (\Delta Z_2^2) \;]$

$$= 2 \; tr[Z_2^2 Z_1 \Delta Z_1 + Z_1^2 Z_2 \Delta Z_2] \tag{49}$$

$$\begin{aligned} Again, \Delta Z_1 = -[\; & \Delta Y_1^H Y_1 + Y_1^H \Delta Y_1 + \bar{\Lambda}_1 \Delta Y_1^H Y_1 \bar{\Lambda}_1' \\ & + \bar{\Lambda}_1 Y_1^H \Delta Y_1 \bar{\Lambda}_1']. \end{aligned} \tag{50}$$

$$\begin{aligned} and, \Delta Z_2 = -[\; & \Delta Y_1^H Y_1 + Y_1^H \Delta Y_1 + \Lambda_1' \Delta Y_1^H Y_1 \Lambda_1 \\ & + \Lambda_1' Y_1^H \Delta Y_1 \Lambda_1]. \end{aligned} \tag{51}$$

Substituting Eqs. (51) and (50) in equation (49) we get:

$$\Delta J_1 = -2tr[Y_1(Z_1^2 Z_2 + Z_2^2 Z_1 \Lambda_1 Z_1^2 Z_2 \Lambda_1' + \bar{\Lambda}_1' Z_2^2 Z_1 \bar{\Lambda}_1) \Delta Y_1^H$$

$$+ (Z_1^2 Z_2 + Z_2^2 Z_1 + \Lambda_1 Z_1^2 Z_2 \Lambda_1' + \bar{\Lambda}_1' Z_2^2 Z_1 \bar{\Lambda}_1) Y_1^H \Delta Y_1]$$

$$\begin{aligned} \therefore \Delta J_1 &= -\, 2tr[Y_1 Z_3 \Delta Y_1^H + Z_3 Y_1^H \Delta Y_1] \\ &= -\, 2tr[Y_1 Z_3 \Delta Y_1^H] - 2tr[Z_3 Y_1^H \Delta Y_1]. \end{aligned} \tag{52}$$

We will now show that each of the terms $tr[Z_3 Y_1^H \Delta Y_1]$ and $tr[Y_1 Z_3 \Delta Y_1^H]$ can be expressed in terms of the quantities $U_1, \Gamma_1, X_1 \; and \; B$.

First, consider $tr[Z_3 Y_1^H \Delta Y_1]$.

$$\begin{aligned} \text{Recall, } Z &= (M X_1 \Lambda_1)^T Y_1 \Lambda_1' + (K X_1)^T Y_1 \\ &= P^T Y_1 \Lambda_1' + Q^T Y_1 \\ &= P^T \bar{P} \Psi_1 + Q^T \bar{Q} \; \Psi_1 \end{aligned}$$

$$\text{Thus, } Z = C_1 \; \Psi_1. \tag{53}$$

$$\text{Hence, } \Delta \Psi_1 = C_1^{-1} \Delta Z \tag{54}$$

$$Thus, \Delta Y_1 = -K\bar{X}_1 \Delta\Psi_1 = -K\bar{X}_1 C_1^{-1}\Delta Z \tag{55}$$

So, $tr[Z_3 Y_1^H \Delta Y_1] = -tr[Z_3 Y_1^H K\bar{X}_1 C_1^{-1}\Delta Z]$

Now since Z satisfies the Sylvester equation

$$\Lambda_1 \ \mathbb{Z} - \mathbb{Z} \ \Lambda_1' = \ - \ \Lambda_1 X_1^T B\Gamma,$$

we have,

$$\Lambda_1 \ (\Delta Z) - (\Delta Z) \ \Lambda_1' = - \ \Lambda_1 X_1^T B(\Delta\Gamma). \tag{56}$$

Also the analytical solution of the above Sylvester equation can be written as:

$$\Delta Z = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \ \gamma_{jk} \ (\Lambda_1)^j \ (-\Lambda_1 X_1^T B \ \Delta\Gamma \ ) \ (\Lambda_1')^k. \tag{57}$$

$$\begin{aligned} &So, \ tr[Z_3 Y_1^H \Delta Y_1] \\ &= -tr[Z_3 Y_1^H K\bar{X}_1 C_1^{-1}\Delta Z], \\ &= -tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \ \gamma_{jk} \ Z_4(\Lambda_1)^j \ (-Z_5) \ (\Lambda_1')^k], \\ &= tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \ \gamma_{jk} \ (\Lambda_1')^k Z_4(\Lambda_1)^j \ Z_5]. \end{aligned} \tag{58}$$

where, $Z_4 = Z_3 Y_1^H K\bar{X}_1 C_1^{-1}$ and $Z_5 = \Lambda_1 X_1^T B \ \Delta\Gamma$.

Since U_1 satisfies the Sylvester equation

$\Lambda_1' \ U_1 - \ U_1 \ \Lambda_1 = - \ Z_3 Y_1^H K\bar{X}_1 C_1^{-1}$,

we can write

$$U_1 = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \ \gamma_{jk} \ (\Lambda_1')^k Z_4(\Lambda_1)^j. \tag{59}$$

So, finally we have;

$$tr[Z_3 Y_1^H \Delta Y_1] = tr[U_1 \Lambda_1 X_1^T B\Delta\Gamma]. \tag{60}$$

Next, consider $tr[Y_1 Z_3 \Delta Y_1^H]$.

$$\begin{aligned} & tr[Y_1 Z_3 \Delta Y_1^H] \\ & = tr[Y_1 Z_3 (\Delta Y_1)^H], \\ & = -tr[Y_1 Z_3 (\Delta Z)^H (C_1^{-1})^H X_1^T K], \quad (using\ 55) \\ & = -tr[(C_1^{-1})^H X_1^T K Y_1 Z_3 (\Delta Z)^H]. \end{aligned} \tag{61}$$

$Since, \quad \Lambda_1\ (\Delta Z) - (\Delta Z)\ \Lambda'_1 = -\ Z_5,$

We have :

$\bar{\Lambda}'_1 (\Delta Z)^H - (\Delta Z)^H \bar{\Lambda}_1 = (Z_5)^H.$

Also the solution $(\Delta Z)^H$ of the above Sylvester equation can be written as

$$(\Delta Z)^H = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk} (\bar{\Lambda}'_1)^j\ Z_5^H\ (\bar{\Lambda}_1)^k. \tag{62}$$

Let $Z_6 = (C_1^{-1})^H X_1^T K Y_1 Z_3$.

$$\begin{aligned} & Then\ tr[Y_1 Z_3 \Delta Y_1^H] \\ & = -tr[Z_6 (\Delta Z)^H], \\ & = -tr[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk}\ Z_6\ (\bar{\Lambda}'_1)^j\ \{Z_5^H\}\ (\bar{\Lambda}_1)^k], \\ & = -tr[\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \delta_{jk} (\bar{\Lambda}_1)^k \{Z_6\} (\bar{\Lambda}'_1)^j\ Z_5^H]. \end{aligned} \tag{63}$$

Now, define the matrix U_2 to be the unique solution of the Sylvester equation

$$\bar{\Lambda}_1 \mathbb{U}_2 - \mathbb{U}_2 \bar{\Lambda}'_1 = -Z_6. \tag{64}$$

$$\begin{aligned} \text{Then, } & tr[Y_1 Z_3 \Delta Y_1^H], \\ & = -tr[U_2 (\Delta\Gamma)^H B^H \bar{X}_1 \bar{\Lambda}_1], \\ & = -tr[B^H \bar{X}_1 \bar{\Lambda}_1 U_2 (\Delta\Gamma)^H]. \end{aligned} \tag{65}$$

Thus by Eq. (52)

$$\Delta J_1 = 2tr[U_1 \Lambda_1 X_1^T B \Delta\Gamma + B^H \bar{X}_1 \bar{\Lambda}_1 U_2 (\Delta\Gamma)^H], \tag{66}$$

which gives us

$$\nabla_\Gamma (J_1) = 2[U_1 \Lambda_1 X_1^T B]^T. \tag{67}$$

The above results lead to the following algorithm.

Algorithm 1: *A Robust Partial Quadratic Eigenvalue Assignment Algorithm*

Inputs:

(i) The matrices M, D, and K; $M > 0,\ K = K^T,\ D = D^T$.

(ii) The control matrix B of order $n \times m$ $(n \leq m)$.

(iii) A self-conjugate set of complex numbers $\{\mu_1, \ldots, \mu_p\}$.

(iv) Tolerance $\in$ and maximum number of iterations, Max_{iter}.

***Output** :*

Real feedback matrices F and G such that the eigenvector matrix Y of the closed loop pencil has improved condition number.

Step 0: Form the matrices $\Lambda_1,\ \Lambda_1',\ X_1,\ C_1$. Set k = 1.

Step 1: Choose a matrix $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \in C^{m\times p}$ such that whenever $\mu_j = \bar{\mu}_k$, $\gamma_j = \bar{\gamma}_k$.

Step 2: Compute the unique solution Z of the Sylvester equation $\Lambda_1 Z - Z\Lambda_1' = -\ \Lambda_1 X_1^T B\Gamma$. If cond(Z) is large, take another Γ.

Step 3: Compute Y_1 and the solution U_1 of the Sylvester equation: $\Lambda_1'\ U_1 - U_1\ \Lambda_1 = \ \ Z_3 Y_1^H K \bar{X}_1 C_1^{-1}$.

Step 4: Compute $Grad = \bigtriangledown_\Gamma(J_1)$. If $\|Grad\|_F\ < \epsilon$ or if the number of iterations exceed Max_{iter}, go to step 5. Else, go to Step 4 .

Step 4: Compute a new Γ using a gradient based optimization method, set $k = k + 1$ and repeat from Step 2.

Step 5: Record the minimum value obtained for J_1 and corresponding value of Γ . For this Γ compute the matrices F and G using formulas in Theorem 4.1. Stop.

Computation of new Γ *in Step 4:*

The function to be minimized is $J_1 = \left\|(I_p - Y_1^H Y_1 - \Lambda_1' Y_1^H Y_1 \Lambda_1)^2\right\|_F^2$. Here Γ is a parameter and Y_1 is a function of Γ. We denote the current Γ by Γ_{old} and the new Γ by Γ_{new}. Then Γ_{new} can be obtained as follows:

i) Replace Γ_{old} by $\hat{\Gamma} = \Gamma_{old} + \alpha\ d_j$ where d_j is given by $d_j = \ -\ D_j\ Grad$. Here Grad represents the current gradient, D_j is the *metric* obtained as in the BFGS method and α is a scaler.

ii) Obtain the value $\widehat{Y}_1$ of Y_1 corresponding to $\widehat{\Gamma}$, as follows:

Let $\widehat{Y}_1 = \{\widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_p\}$ and $\widehat{\Gamma} = \{\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_p\}$ then obtain $\widehat{y}_i$ by solving the Sylvester equation $(M\mu_i^2 + D\mu_i + K)\ \widehat{y}_i = B\ \widehat{\gamma}_i$ for $i = 1 : p$.

iii) Find $l = \min_\alpha \left\|(I_p - \widehat{Y}_1^H \widehat{Y}_1 - \Lambda_1' \widehat{Y}_1^H \widehat{Y}_1 \Lambda_1)^2\right\|_F^2$. (This is obtained by using the MATLAB function *fminbnd*).

iv) $\Gamma_{new} = \Gamma_{old} + l\ d_j$.

Efficiency:

The cost of the algorithm is dominated by solution of the Sylvester equation in Step 2 and the gradient evaluation of the Gradient in Step 3. The flop-counts for these two steps are of $O(p^3)$ only.

7 A gradient based method for simultaneous improvement of feedback norms and condition number of eigenvector matrix

We have so far discussed feedback norm minimization and the minimization of the conditioning of the closed loop eigenvector matrix separately. The question naturally arises whether these two aspects can be combined in one setting. That is, can the feedback norms and the condition number of the closed loop eigenvector matrix be improved simultaneously? This problem has been considered by Varga (see Ref [5]) in the case of the first order model. Following Varga, we now consider the following objective function for the quadratic pencil:

$$\mathbb{O} = \frac{[(C1)(\alpha)\mathbb{I} + (C2)(1-\alpha)\mathbb{J}_1]}{[(C1)(\alpha) + (C2)(1-\alpha)]}, \tag{68}$$

where $\mathbb{I} = \dfrac{1}{2}[\|S\|_F^2 = \frac{1}{2}\,[\|F\|_F^2 + \quad \|G\|_F^2]$ and $J_1 = \left\|(I_p - Y_1^H Y_1 - \Lambda_1' Y_1^H Y_1 \Lambda_1)^2\right\|_F^2$. and C1, C2 and α are constants. Note that when $\alpha = 1$, we have the minimum-norm problem and when $\alpha = 0$, we have the robust problem.

The gradient of $\mathbb{O}$ with respect to Γ can be computed as follows (see Ref [23] for details):

$$\bigtriangledown_\Gamma(\mathbb{O}) = \frac{[(C1)(\alpha)\bigtriangledown_\Gamma(\mathbb{I}) + (C2)(1-\alpha)\bigtriangledown_\Gamma(\mathbb{J}_1)]}{[(C1)(\alpha) + (C2)(1-\alpha)]} \tag{69}$$

$\bigtriangledown_\Gamma(\mathbb{I})$ is given by Theorem 5.2 and $\bigtriangledown_\Gamma(\mathbb{J}_1)$ is given by Theorem 6.1, respectively. If the magnitudes of the elements of $\bigtriangledown_\Gamma(\mathbb{I})$ and $\bigtriangledown_\Gamma(\mathbb{J}_1)$ are widely disparate then the constants C1 and C2 need to be chosen to scale their magnitudes, to prevent one gradient totally dominating the other in the calculation of the gradient of $\mathbb{O}$. After the constants C1 and C2 have been chosen appropriately for a particular problem, the value of the constant α is varied between 0 and 1 to bring about different amounts of reduction in the feedback norms and the condition number of the eigenvector matrix.

8 Results of Numerical Experiments:

The results on our numerical experiments are presented in tabular format below. The problems are taken from Refs. [2,24]. Table 1, Table 2, Table 3,

Table 4, Table 5 contain, respectively, the results of Problem 1, Problem 2 (i), Problem 2(ii), Problem 2 (iii), and Problem 3. Table 6 contains results of comparisons of our Algorithm 1 with those of the Qian-Xu method. Some of these results are also presented in graphic form in Figure 1. In the following, the percentage reduction in the Frobenius norm is calculated as follows:

Percentage reduction in the Frobenius norm $= 100 * \dfrac{\text{IN} - \text{FN}}{\text{IN}}$

where IN = Value of the norm with initial Γ and

FN = Value of the norm with the final Γ.

The **Percentage Reduction in the Condition Number** is similarly defined.

Accuracy= The Frobenius norm of the difference between the desired closed-loop eigenvalues and the actual closed-loop eigenvalues obtained for optimal Γ.

Verifying the Sensitivity of the Closed-loop Eigenvalues under Small Perturbations:

To verify the robustness of our solutions to the partial eigenvalue assignment problem, we perturb the stiffness matrix K and then compute the closed loop eigenvalues corresponding to the feedback matrices F and G (obtained for the optimal Γ for the unperturbed problem) keeping M, D and B unchanged. The rationale for leaving the mass and stiffness matrices unperturbed is that the mass matrix is usually accurately determined and the damping matrix is hard to estimate in practice. On the other hand, the stiffness matrix K is often not determined accurately. We then calculate the norm (**Accuracy with perturbed K**) of the difference of the matrices of the closed loop eigenvalues under perturbation obtained as described and the corresponding closed loop eigenvalues actually obtained for the unperturbed problem.

Problem 1:

$$\mathbf{M} = \begin{pmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876 \end{pmatrix}$$

$$\mathbf{K} = \begin{pmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0.3450 & 0.4578 \\ 0.0579 & 0.7630 \\ 0.5967 & 0.9990 \\ 0.2853 & 0.3063 \end{pmatrix}$$

The open-loop eigenvalues are:

$\{-0.0861 \pm 1.6242i, -0.1748 \pm 1.1922i, -0.4480 \pm 0.2465i, -0.1022 \pm 0.8876i\}$.

The first two eigenvalues $\{-0.0861 \pm 1.6242i\}$ were reassigned to $\{-8 \pm 1.6242i\}$, the other eigenvalues were kept unchanged.

Problem 2:

$\mathbf{M} = 4 * \mathbf{I_{n\times n}}$, $\mathbf{D} = 4 * \mathbf{I_{n\times n}}$

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

(Case i) With $n = 8$, there are 16 open-loop eigenvalues of which the first four,viz., $\{-0.5 \pm 0.8438i, -0.5 \pm 0.7769i\}$ are reassigned to $\{-3 \pm 0.8438i, -5 \pm 0.7769i\}$, keeping other eigenvalues unchanged.

(Case ii) With $n = 10$, there are 20 open-loop eigenvalues of which the first six, viz., $\{-0.5 \pm 0.7375i, -0.5 \pm 0.8518i, -0.5 \pm 0.8090i\}$ are reassigned to

$\{-8\pm.7375i, -8\pm0.8518i, -8\pm0.8090\}$, keeping other eigenvalues unchanged.

(Case iii) With $n = 200$, there are 400 open-loop eigenvalues of which the first six, viz., $\{-0.5 \pm 0.8660i, -0.5 \pm 0.8659i, -0.5 \pm 0.8657i\}$ are reassigned to $\{-1 \pm 0.8660i, -1 \pm 0.8659i, -1 \pm 0.8657i\}$, keeping other eigenvalues unchanged.

Problem 3:

$$\mathbf{K} = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{M} = 10I_{3,3}, \mathbf{D} = O_{3,3}$$

The open-loop eigenvalues are $\{\pm3.6039i, \pm2.49399i, \pm0.8901i\}$. The first two eigenvalues $\{\pm3.6039i\}$ were reassigned to $\{-1, -2\}$, the other eigenvalues were kept unchanged.

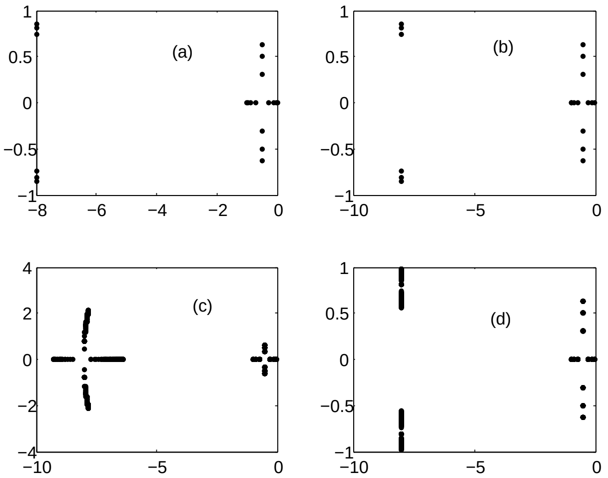

Fig. 1. Closed loop eigenvalues for problem 2(ii):(a) Desired eigenvalues, (b) Eigenvalues for optimal Γ for unperturbed problem, (c) Eigenvalues for the Γ in case (b) under 100 random perturbations of order 0.01, (d) Eigenvalues for the Γ in case (b) under 100 random perturbations of order 0.001

Note on convergence of Algorithm 1:

For the five problems the optimal Γ was obtained after 2, 2, 8, 2 and 5 sweeps respectively. We set $\epsilon = 10^{-4}$ for problems 1 and 2(i) and $\epsilon = 10^{-8}$ for the remaining three problems. In most cases the algorithm stopped when the number of iterations equalled Max_{iter}. The solution is dependent on the choice of the initial Γ and the initial metric D_0 . For some choices of Γ and D_0 the condition number did not decrease.

Comparison of algorithm 1 with the Qian-Xu method

We now present the results obtained by using the Qian-Xu method for Problems 1 and 2. Note this method is for condition number reduction only.

Table 6: Results of the Qian-Xu method

A comparison of the two methods show that they produce almost the same accuracy, and condition number reduction in both cases are comparable. However since our method is optimization based, this method is also suitable for large-scale computations using specialized large scale optimization techniques. Furthermore our method can also handle simultaneous minimization of both feedback norms and the closed-loop eigenvector conditioning. It may be noted in the context of comparison that the "accuracy" measure is computed differently in the Quain-Xu paper.

Conclusion

The problem of designing a robust active controller for a vibrating structure modeled by a system of second-order matrix differential equations is the one in which a feedback controller has to be constructed such that the feedback matrices have minimum norms and the condition number of the closed-loop eigenvector matrix is as small as possible to ensure that the closed loop eigenvalues are not sensitive to small perturbations of the data. Mathematically, this leads to minimum-norm and robust partial quadratic eigenvalue assignment problems (MNPQEVAP and RPQEVAP). Basically these problems are optimization problems, and one special advantage of solving these problems in an optimization setting is that an excellent numerical optimization technique, such as the BFGS method, can be profitably used. However, a bottleneck in using this technique is deriving parametric expressions for feedback matrices and developing appropriate gradient formulas. In case of the problems under consideration here, a further computational challenge is to develop such gradient formulas using only a few eigenvalues and the corresponding eigenvectors of the associated quadratic eigenvalue problem, since in practice it is impossible to compute all the eigenvalues and eigenvectors of a large quadratic matrix pencil even with the state-of-the-art computational techniques. In the present paper and in another recent one, (i) parametric expressions for feedback matrices via Sylvester equations have been derived, and (ii) appropriate gradient formulas both for minimum-norm and robust eigenvalue assignment problems have been developed using only the small number of eigenvalues that need to be reassigned and the associated eigenvectors. These techniques are, therefore, implementable in practice even for large-scale structures. However, some more work still needs to be done. One of the underlying mathematical problems is how to choose the initial parametric matrix in each such algorithm so that convergence can be guaranteed within a reasonable number of steps. It is to be noted in this context that the underlying optimization problems are difficult and further research is needed and currently underway for finding local and global solutions.

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References

[1] B.N.Datta, D.R.Sarkissian, Theory and computations of some inverse eigenvalue problems for the quadratic pencil. *Contemporary Mathematics,* Vol. 280, 2001, pp. 221-240.

[2] B.N.Datta, Sylvan Elhay, Y.Ram, Orthogonality and partial pole assignment

for the symmetric definite quadratic pencil. *Linear Algebra and its Applications*, Vol. 257, 1997, pp. 29-48.

[3] F.Tisseur, K.Meerbergen, The quadratic eigenvalue problem. *SIAM Review,* Vol. 43, No.2, pp. 235-286.

[4] B.N.Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Elsevier Academic Press, Boston, 2003.

[5] A.Varga, A numerically reliable approach to robust pole assignment for descriptor systems. *Future Generation Computer Systems*, Vol. 19, 2003, pp. 1221- 1230.

[6] S.P.Bhattacharyya, E.deSouza, Pole assignment via Sylvester's equation. *Systems and Control Letters*, Vol. 1, 1982, pp. 261-283.

[7] L.H.Keel, J.A.Fleming, S.P.Bhattacharyya, Minimum norm pole assignment via Sylvester's equations. *Contemporary Mathematics*, Vol. 47, 1985, pp. 265-272.

[8] R.K.Cavin III, S.P.Bhattacharyya, Robust and well-conditioned eigenstructure assignment via Sylvester's equation. *Optimal Control Application and Methods,* Vol. 4, 1983, pp. 205-212.

[9] B.N.Datta, D.R.Sarkissian, Partial eigenvalue assignment in linear systems: existence, uniqueness and numerical solution. *Proceedings of the Mathematical Theory of Networks and Systems (MTNS)*, Notre Dame, August 2002.

[10] G.L.Sleijpen, H.A.van der Vorst, Z. Bai, Jacobi-Davidson algorithms for various eigenproblems: A working document, H.A. van der Vorst's homepage, 1999, available at www. math.uu.nl/people/vorst.

[11] D.J.Inman, *Vibration, with Control, Measurement and Stability.* Prentice Hall, Englewood Cliffs, NJ (1989).

[12] J.Nocedal, Stephen Wright, *Numerical Optimization* , 2nd ed., Springer, New York, 2006.

[13] M.Bazaraa, H.Sherali, C.Shetty, *Nonlinear Programming: Theory and Algorithms*, 2nd ed., John Wiley and Sons, New York, 1993.

[14] S.Brahma, B.N.Datta, A norm-minimizing parametric algorithm for quadratic partial eigenvalue assignment via Sylvester equation. *Proceedings of the IEEE European Control Conference*, Kos, Greece, July 2007, pp. 490-496.

[15] S.Brahma and B.N.Datta, A Sylvester-equation based parametric approach for minimum norm and robust partial quadratic eigenvalue assignment problems. *Proceedings of the IEEE Mediterranean Control Conference*, Athens, Greece, June 2007, pp. 1-6.

[16] Y.M.Ram, S.Elhay, Pole assignment in vibrating systems by multi-input control. *Journal of Sound and Vibration*, Vol. 230, 2000, pp. 309-321.

[17] J.Qia‘n, S.Xu, Robust partial eigenvalue assignment problem for the second-order system. *Journal of Sound and Vibration*, Vol. 282, 2005, pp. 937-948

[18] B.N.Datta, W. W.Lin, J.N.Wang, Robust partial pole assignment for vibrating structures with aerodynamic effect. *IEEE Transactions on Automatic Control,* Vol. 51, No. 12, 2006, pp. 1979-1984.

[19] B.N.Datta, W.W.Lin, and J.N.Wang, Robust and minimum gain partial pole assignment for a third order system. *Proceedings of the IEEE Conference on*

Decision and Control, Hawaii, USA, Dec 2003, pp. 2358-2363.

[20] B.N.Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Elsevier Academic Press, Boston, 2003.

[21] B.N.Datta, S.Elhay, Y.Ram, An algorithm for the partial multi-input pole assignment of a second-order control system. *Proceedings of the IEEE Conference on Decision and Control*, Kobe, Japan, December 1996, pp. 2025-2029.

[22] R.Horn and C.Johnson, *Matrix Analysis.* Cambridge University Press, New York, 1991.

[23] S.K.Brahma, *Robust and Minimum Norm Partial Quadratic Eigenvalue Assignment Problems: Theory and Computations.* Ph.D. Dissertation, Department of Mathematical Sciences, Northern Illinois University, IL 2006.

[24] P.Benner, A.J.Laub, V.Mehrmann, A collection of bench mark examples for the numerical solution of algebraic Ricatti equations I: continuous time case, Technical Report SPC 95 _ 23 , TU Chemnitz-Zwickau, December 1995.

[25] N.K. Nichols, J. Kautsky, Robust eigenstructure assignment in quadratic matrix polynomials: nonsingular case. *SIAM Journal on Matrix Analysis and Applications*, Vol. 23, No. 1, 2001, pp. 77–102.

[26] B.Porter and R.Crossley, *Modal Control: Theory and Applications* , Barnes & Noble, New York, 1972.

[27] Y.Saad, A projection method for partial pole assignment in linear state feedback. *IEEE Transactions on Automatic Control,* Vol. 33, No 3, 1998, pp.

290–297.

[28] B.N.Datta and Y. Saad, Arnoldi methods for large sylvester-like observer matrix equations and an associated algorithm for partial spectrum assignment, *Linear Algebra and its Applications*, Vol. 154-156, 1991, pp. 225–244.

[29] J. Carvalho and B. N. Datta, A new algorithm for generalized Sylvester equation and its applications to state and velocity estimations in vibrating systems. To appear in Numerical Linear Algebra with Applications.

[30] B. N. Datta and D. R. Sarkissian, Feedback control in distributed paprameter gyroscopic systems : A solution of the partial eigenvalue assignment. *Mechnaical Systems and Signal Processing*, Vol. 16, No. 1, 2002, pp. 3–17.

Table 1: Numerical Results for Problem 1

Results of the three algorithms for Problem 1			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage Reduction	Norm of ΔK	Accuracy with perturbed K
(i) Norm Reduction(I)	$O(10^{-14})$	99.99	0.3181	0.1687
(ii) Condition Number Reduction(CN) (Algorithm 1)	$O(10^{-14})$	99.74	0.0172 0.0024	0.0109 0.0015
(iii) Simultaneous Reduction of Norms and Condition Numbers for $\alpha = 0.4, C1 = 50,\ C2 = 0.5$	$O(10^{-13})$	$99.99(I)$ $99.28(CN)$	$3.0672e-004$ $3.5702e-005$ $2.8591e-006$	$1.9691e-004$ $2.2926e-005$ $1.8360e-006$

Table 2: Numerical results for problem 2(i)

Results of the three algorithms for Problem 2(i)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage Reduction	Norm of ΔK	Accuracy with perturbed K
(i) Norm Reduction(I)	$O(10^{-14})$	99.94	0.1101	0.0312
(ii) Condition Number Reduction(CN) (Algorithm 1)	$O(10^{-14})$	99.66	0.0253	0.0072
			0.0058	0.0017
(iii) Simultaneous Reduction of Norms and Condition Numbers for $\alpha = 0.4, C1 = 10^{16}, C2 = 1$	$O(10^{-14})$	$99.98(I)$	$5.7261e-004$	$1.6406e-004$
		$98.68(CN)$	$2.5621e-005$	$7.3413e-006$
			$5.5808e-006$	$1.5991e-006$

Table 3: Numerical results for problem 2(ii)

Results of the three algorithms for Problem 2(ii)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage Reduction	Norm of ΔK	Accuracy with perturbed K
(i) Norm Reduction(I)	$O(10^{-6})$	99.92	$1.9438e-001$	$7.4819e+000$
(ii) Condition Number Reduction(CN) (Algorithm 1)	$O(10^{-5})$	99.39	$1.4198e-002$	$2.8098e+000$
			$1.0908e-004$	$4.5488e-001$
(iii) Simultaneous Reduction of Norms and Condition Numbers for $\alpha = 0.4, C1 = 10^9,\ C2 = 1$	$O(10^{-5})$	$99.99(I)$ $98.94(CN)$	$3.3279e-005$	$2.3029e-001$
			$2.9898e-006$	$2.5192e-002$
			$6.0432e-007$	$5.5007e-003$

Table 4: Numerical results for problem 2(iii)

Results of the three algorithms for Problem 2(iii)			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage Reduction	Norm of ΔK	Accuracy with perturbed K
(i) Norm Reduction(I)	$O(10^{-2})$	99.97	$3.1643e+000$	$1.2004e+000$
(ii) Condition Number Reduction(CN) (Algorithm 1)	$O(10^{2})$	99.32	$1.4159e-001$	$3.1010e-001$
			$3.0841e-002$	$2.2944e-001$
(iii) Simultaneous Reduction of Norms and Condition Numbers for $\alpha = 0.4, C1 = 10^{17},\ C2 = 1$	$O(10^{-2})$	$99.99(I)$ $96.89(CN)$	$1.2178e-003$	$1.0295e-001$
			$2.8063e-004$	$9.2421e-002$
			$2.0837e-005$	$1.3234e-001$

Table 5: Numerical results for problem 3

Results of the three algorithms for Problem 3			Perturbation results for Algorithm 1	
Algorithm	Accuracy	Percentage Reduction	Norm of ΔK	Accuracy with perturbed K
(i) Norm Reduction(I)	$O(10^{-14})$	99.79	0.0693	0.0143
(ii) Condition Number Reduction(CN) (Algorithm 1)	$O(10^{-13})$	99.96	0.0182	0.0037
			0.0015	0.0003
(iii) Simultaneous Reduction of Norms and Condition Numbers for $\alpha = 0.4, C1 = 10^7,\ C2 = 1$	$O(10^{-15})$	$98.93(I)$	$2.6739e-004$	$5.4816e-005$
		$97.86(CN)$	$2.2863e-005$	$4.6869e-006$
			$5.5511e-009$	$1.1379e-009$

Table 6: Numerical results of the Qian-Xu Method

Prob.	Percentage Reduction	Accuracy
1	97.5151	$5.8042e-010$
2(i)	99.61	1.8195
2(ii)	98.56	$8.3242e-008$
2(iii)	98.14	0.0525
3	99.45	$2.1915e-014$