A Sylvester-Equation Based Parametric Approach for Minimum Norm and Robust Partial Quadratic Eigenvalue Assignment Problems

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***Abstract*— The Partial Quadratic Eigenvalue Assignment Problem is the problem of reassigning a small number of undesirable eigenvalues of a quadratic matrix pencil using feedback. The problem arises in controlling resonance in vibrating structures and also in stabilizing control systems. The solution involves the computation of a pair of feedback matrices. For practical effectiveness, the magnitudes of the feedback norms need to be reduced and the conditioning of the closed-loop eigenvalues needs to be improved.**

In this paper, we propose new optimization methods for solving these problems. An important practical aspect of these methods is that the gradient formulas needed to solve the underlying unconstrained optimization problems are computed using only a small number of eigenvalues and eigenvectors of the quadratic pencil, which are all that can be computed using the state-of-the-art computational techniques.

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I. Introduction

It its well-known that vibrating structures, such as bridges, highways, buildings, air and space crafts, etc., can be modeled by a system of second-order differential equation of the form:

$$\mathbf{M}\ \ddot{\mathbf{q}}(\mathbf{t}) + \mathbf{D}\ \dot{\mathbf{q}}(\mathbf{t}) + \mathbf{K}\ \mathbf{q}(\mathbf{t}) = \mathbf{f}(\mathbf{t}) \tag{1}$$

where M, D, K are constant real $(n \times n)$ matrices, q(t) and f(t) are real n vectors. M is called the *mass matrix*, D the *damping matrix* and K the *stiffness matrix*, f(t) represents an external force or an applied force, t represents time and n an integer called the dimension of the system.

In most applications, the matrices M, K and D are symmetric, furthermore, M is positive definite and K positive semi-definite.

The dynamics of the structures modeled by equation (1) are governed by the eigenvalues and eigenvectors of the quadratic matrix polynomial ([12], [10]) $P(\lambda) \equiv M\lambda^2 + D\lambda + K$. If M is non-singular, $P(\lambda)$ has 2n eigenvalues which are the roots of the equation $det(P(\lambda)) = 0$.

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The **natural frequencies** of the structure are related to the eigenvalues of $P(\lambda)$ and the **mode shapes** are just the eigenvectors.

When a natural frequency becomes equal or close to a frequency of the external force acting upon the structure, such as, the earthquake, gusty winds or weights of the human bodies, a dangerous situation, called resonance, occurs causing partial or complete destruction of the structure.

A traditional approach to combat resonance is by the use of passive damping. The damping is adjusted in an ad hoc way so that resonant frequencies are eliminated. A more systematic and mathematical approach is to use active controllers. In fact, in several countries like Japan, which are prone to earthquakes, buildings have been constructed with active controllers. The idea behind the active controllers is to apply a control force of the form: $f(t) = Bu(t)$, where B is a given real $n \times m$ matrix $(m \leq n)$ and $u(t)$ is a real vector given by

$$u(t) = F^T\ \dot{q}(t) + G^T\ q(t). \tag{2}$$

Here, q(t) and q̇(t) are assumed to be known. F and G are constant real $(n \times m)$ matrices, to be determined, called the *feedback matrices.* From (1) and (3), we obtain the equation

$$P_c(\lambda) \equiv M\ddot{q}(t) + (D - B\ F^T)\dot{q}(t) + (K - B\ G^T)q(t) = 0 \tag{3}$$

The feedback matrices F and G are now determined such that from the spectrum of the associated closed-loop matrix pencil

$$P_c(\lambda) \equiv \lambda^2 M\ + \lambda(D - B\ F^T)\ + (K - B\ G^T)\ = 0 \tag{4}$$

is the one that is obtained from the spectrum of $P(\lambda)$ by replacing the resonant or unstable eigenvalues with suitably chosen ones while keeping the remaining eigenvalues and associated eigenvectors the same. *The last property will guarantee that no spurious modes will occur into the frequency range of interest, which is a major concern for vibration engineers while modifying the coefficient matrices in this way.*

The problem of finding F and G, is referred to as the ***partial quadratic eigenvalue assignment problem*** (PQEVAP).
In the multi-input case (i.e. when $m > 1$), the solution of the PQEVAP is not unique. This fact can be exploited by determining F and G in such a way that not only $P_c(\lambda)$ has the desired spectrum but also the system has some additional desirable features, for example, smaller feedback matrices and small condition number of the closed-loop eigenvalues. The smaller norms of the feedback matrices lead to smaller control signals and this in turn leads to lesser energy consumption [16]. The problem of finding the feedback matrices such that their norms are as small as possible is known as the ***minimum norm partial quadratic eigenvalue assignment problem*** (MNPQEVAP). Improving the conditioning of the closed-loop eigenvalues ensures that small changes in the data matrices do not lead to drastic changes in the closed-loop eigenvalues. The problem of finding the feedback matrices such that the conditioning of the closed-loop eigenvalues is as good as possible is called the ***Robust Partial Quadratic Eigenvalue Assignment Problem (RPQEVAP)*** One way to solve these problems is to obtain a parametric family of feedback matrices and then choose the parametric matrix in a proper way to solve the problem under consideration. A parametric Sylvester-equation approach was first developed in [3]for the complete pole assignment problem in the standard first-order state space form and using this, minimum-norm and robust pole assignment problems in first-order state space form have been solved in [6], [16], [21]. In a recent paper [5], the authors have generalized this to solve MNPQEVAP in quadratic setting, that is, without requiring any transformation to the standard first-order form. Note that computationally, such a transformation is not desirable, because it might need inversion of the ill-conditioned mass matrix and furthermore, the nice structures offered by a practical problem, such as the symmetry, sparsity, definiteness, etc., which are often assets for large-scale computations, are totally destroyed. The present paper deals with RPQEVAP and also the composite case of simultaneous reduction of the feedback norms and the condition number of the closed-loop eigenvector matrix. Both MNPQEVAP and RPQEVAP are basically optimization problems. Specifically, they are nonlinear unconstrained minimization problems. It is natural to solve these problems in an optimization setting. An advantage of doing so is that one can make use of some of the excellent numerical optimization methods available now in optimization literature [1]. A challenge to use an existing optimization technique is, however, to develop a suitable gradient formula, needed for implementations. For MNPQEVAP and RPQEVAP, ad additional challenge is to develop such gradient formulas in terms of only a limited small number of eigenvalues and eigenvectors of the associated quadratic matrix eigenvalue problems that only can be computed or measured, since it is not possible to compute the whole spectrum and the eigenvectors of a large quadratic matrix pencil even using the state-of-the-art matrix computational techniques[7,20]. Such computable gradient formulas are provided in Theorem 2, and Theorem 5, for MNPQEVAP and RPQEVAP, respectively and in Section VI for the composite case. The other papers dealing with parametric expressions for PEVAP and PQEVAP in quadratic setting itself include, respectively , [11, 19] and [9,10,12,and 18]. But none of these papers address the minimum norm and robustness issues. The only paper dealing with robustness issue for PQEVAP published so far is the paper [17]. The method in this paper, unlike ours,is based on eigenvector selections and is not optimization-based. It is also not always guaranteed to converge. The results on test problems of our algorithm for RPQEVAP are comparable with those of the eigenvector selection method in [17].

II. NOTATIONS:

The following notations are established :
$\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$
$=$ the matrix of eigenvalues of the open-loop pencil $P(\lambda)$,
$\mathbf{\Lambda_1} = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$
$=$ the matrix of open-loop eigenvalues to be reassigned,
$\mathbf{\Lambda_2} = diag(\lambda_{p+1}, \ldots, \lambda_{2n})$
$=$ the matrix of open-loop eigenvalues to remain invariant,
$\mathbf{\Lambda_1'} = diag(\mu_1, \mu_2, \ldots, \mu_p)$
$=$ the matrix of new eigenvalues to replace those in Λ_1,
$\mathbf{X} = (x_1, x_2, \ldots, x_{2n})$
$=$ the matrix of right eigenvectors of the pencil $P(\lambda)$,
$\mathbf{X_1} = (x_1, x_2, \ldots, x_p)$
$=$ the matrix of eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$
$\mathbf{X_2} = (x_{p+1}, \ldots, x_{2n})$
$=$ the matrix of eigenvectors corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$,
$\|\mathbf{X}\|_\mathbf{F} = \ Frobenius\ norm\ of\ X.$
Assumptions: The following assumptions, which are quite reasonable in practice, are made throughout the whole paper.

1) $\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\} \cap \{\mu_1, \mu_2, \ldots, \mu_p\} = \phi$.
2) $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\} = \phi$.
3) $0 \notin \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$

III. A Parametric Expression For The Feedback Matrices

The following theorem provides a parametric solution of the PQEVAP. It is proved in [5]. **Theorem 1 (Parametric Expression for Feedback Matrices)**
Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \in \mathbb{C}^{m \times p}$ be such that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the $(p \times p)$ Sylvester equation

$$\Lambda_1 \mathbb{Z} - \mathbb{Z}\Lambda_1' = -\Lambda_1 X_1^T B \Gamma \tag{5}$$

Then the feedback matrices F and G that solve the PQEVAP are given by $F = M X_1 \Lambda_1 \Phi^T$ and $G = -K X_1 \Phi^T$ where Φ is obtained by solving the linear system $\Phi Z = \Gamma$.

IV. Minimizing The Feedback Norms

Let $\mathbb{I} = \frac{1}{2}\|S\|_F^2 = \frac{1}{2}\ [\|F\|_F^2 + \|G\|_F^2]$ where $S = [G^T\ F^T]$.

Since F and G are both function of the parametric matrix Γ, the MNPQEVAP can be formulated as the problem of finding the matrix Γ for which $\mathbb{I}$ is minimum.
Minimize: $\mathbb{I} = f(\Gamma)$.
The use of a well-known optimization technique, such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, [1], requires gradient evaluation. The following theorem, proved in [4] and [5] shows how to do this evaluation in terms of the known quantities $\Lambda_1, \Lambda_1', X_1$, and B .

Theorem 2 (Gradient Formula for $\mathbb{I}$)
Let $S = [G^T \ F^T]$, $P = MX_1\Lambda_1$, $Q = -KX_1$ and $C = [Q^T \ P^T]$. Let Z satisfy the Sylvester equation:
$\Lambda_1\mathbb{Z} - \mathbb{Z}\Lambda_1' = -\Lambda_1 X_1^T B\Gamma$. If Z is invertible and if U satisfies the Sylvester equation: $\Lambda_1'\mathbb{U} - \mathbb{U}\Lambda_1 = -Z^{-1}CS^H\Phi$ where, $\Phi Z = \Gamma$. Then
$(i)\ S = \Gamma Z^{-1}C$.
$(ii)\ \bigtriangledown_\Gamma(\mathbb{I}) = \frac{1}{2}[Z^{-1}CS^H - U\Lambda_1 X_1^T B]^T$ ■

V. A Gradient Based Method for Robust Partial Quadratic Eigenvalue Assignment

The robust partial quadratic eigenvalue assignment problem can be stated as the problem of finding the feedback matrices F and G, of the closed-loop pencil, such that
i)The closed loop spectrum is the set $\{\mu_1, \mu_2, \ldots, \mu_p, \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\}$
ii) The conditioning of the closed loop eigenvalues is as good as possible.
It is easy to see [4] that the matrix of right eigenvectors of the closed-loop pencil is the matrix Y given by $Y = \begin{pmatrix} Y_1 & X_2 \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{pmatrix}$,
where $Y_1 = [y_1, y_2, \ldots, y_p]$, and y_i is the right eigenvector of the pencil $P_c(\lambda)$ corresponding to the eigenvalue μ_i and y_i satisfies the equation $(M\mu_i^2 + D\mu_i + K)y_i = B\gamma_i$ for $i = 1 : p$. By the Bauer-Fike theorem [7, 8]an overall measure of the conditioning of the eigenvalues of the closed-loop matrix is provided by the condition number of the matrix Y. Now, the conditioning of the eigenvalues of this matrix is best when Y is unitary or orthogonal since in this case the two norm condition number of Y is 1. Thus we seek to determine $\mathbf{\Gamma}$ such that $J = \left\|(I - Y^HY)^2\right\|_F^2$ is as small as possible. This measure of robustness was used before in [6] for the first-order model. It has worked well in the first-order case and also in the quadratic case, as shown by results of our numerical examples here.
If Y were a unitary or orthogonal matrix then

$$\begin{pmatrix} Y_1 \\ Y_1\Lambda_1' \end{pmatrix}^H \cdot \begin{pmatrix} X_2 \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)} \tag{6}$$

or, $Y_1^H X_2 + \bar{\Lambda}_1' Y_1^H X_2\Lambda_2 = O_{(p\times 2n-p)}$.
We now show that J can be split into two parts J_1 and J_2 such parameter matrix Γ. For this, we need the following two theorems, whose proofs are omitted here for space limitations. In any case, they can be found in [4].

Theorem 3 $\begin{pmatrix} -KX_1 \\ MX_1\Lambda_1 \end{pmatrix}^T \cdot \begin{pmatrix} X_2 \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)}$.

■

Theorem 4
If a $(2n \times p)$ matrix $\mathbb{Q}$ satisfies $\mathbb{Q}\begin{pmatrix} X_2 \\ X_2\Lambda_2 \end{pmatrix} = O_{(p\times 2n-p)}$, then there exists a $(p \times p)$ matrix Ψ such that $\mathbb{Q}^T = \begin{pmatrix} -KX_1 \\ MX_1\Lambda_1 \end{pmatrix}.\Psi$

■

From (6) and Theorem 4 we obtain:
$\begin{pmatrix} Y_1 \\ Y_1\Lambda_1' \end{pmatrix} = \begin{pmatrix} -K\bar{X}_1 \\ M\bar{X}_1\Lambda_1 \end{pmatrix}.\Psi_1 = \begin{pmatrix} -K\bar{X}_1\Psi_1 \\ M\bar{X}_1\Lambda_1\Psi_1 \end{pmatrix}$.
Hence, $Y_1 = -K\bar{X}_1\Psi_1 = \bar{Q}\Psi_1$ and
$Y_1\Lambda_1' = M\bar{X}_1\Lambda_1\Psi_1 = \bar{P}\Psi_1$, where $P = MX_1\Lambda_1$ and $Q = -KX_1$.
Let $W_1 = I_p - Y_1^HY_1 - \Lambda_1'Y_1^HY_1\Lambda_1$
and $W_2 = I_{2n-p} - X_2^HX_2 - \bar{\Lambda}_2X_2^HX_2\Lambda_2$. Then by (6) we obtain:
$(I - Y^HY) = \begin{pmatrix} W_1 & O_{(p\times 2n-p)} \\ O_{(2n-p\times p)} & W_2 \end{pmatrix}$.
Therefore, $(I - Y^HY)^2 = \begin{pmatrix} (W_1)^2 & O_{(p\times 2n-p)} \\ O_{(2n-p\times p)} & (W_2)^2 \end{pmatrix}$.
Then $J = \left\|(I - Y^HY)^2\right\|_F^2$
$= \left\|(W_1)^2\right\|_F^2 + \left\|(W_2)^2\right\|_F^2$
$= J_1 + J_2\ (say)$.
Now, the matrix X_2 is independent of Γ, Λ_1', Λ_2 are fixed matrices, and the matrix Y_1 is a function of the parameter Γ. Thus, J_1 is a function of Γ and J_2 is independent of Γ.
So, $\bigtriangledown_\Gamma(J_2) = 0$.
Hence, $\bigtriangledown_\Gamma(J) = \bigtriangledown_\Gamma(J_1)$.
Also, since J_2 remains invariant, therefore, J is as small as possible whenever J_1 is as small as possible. Thus in order to determine $\mathbf{\Gamma}$ for which J is as small as possible, we determine the $\mathbf{\Gamma}$ for which $\mathbf{J_1}$ is as small as possible using the BFGS method. This again requires a gradient formula for J_1 with respect to Γ.

Theorem 5 (Matrix Gradient formula for J_1):
Let $Z_1 \equiv I_p - Y_1^HY_1 - \bar{\Lambda}_1Y_1^HY_1\bar{\Lambda}_1'$,
$Z_2 \equiv I_p - Y_1^HY_1 - \Lambda_1'Y_1^HY_1\Lambda_1$,
$Z_3 \equiv Z_1^2Z_2 + Z_2^2Z_1 + \Lambda_1Z_1^2Z_2\Lambda_1' + \bar{\Lambda}_1'Z_2^2Z_1\bar{\Lambda}_1$,
Let U_1 satisfy the Sylvester equation
$\Lambda_1'\ U_1 -\ U_1\ \Lambda_1 = Z_3Y_1^HK\bar{X}_1C_1^{-1}$. where, $C_1 = P^T\bar{P} + Q^T\bar{Q}$, $P = MX_1\Lambda_1$ and $Q = -KX_1$. Then $\bigtriangledown_{\mathbf{\Gamma}}(\boldsymbol{J_1}) = \mathbf{2[U_1\Lambda_1X_1^TB]^T}$

Proof: From the definition of $J_1 = \left\|(W_1)^2\right\|_F^2$, it follows that:
$J_1 = tr[\{(W_1)^2\}^H(W_1)^2]$
$= tr[(W_1^H)^2(W_1)^2]$.
Thus $J_1 = tr[Z_1^2\ Z_2^2]$.
So, $\Delta J_1 = tr[\ (\Delta Z_1^2)Z_2^2 + Z_1^2(\Delta Z_2^2)\ ]$

$$= 2\ tr[Z_2^2Z_1\Delta Z_1 + Z_1^2Z_2\Delta Z_2] \tag{7}$$

$$Again, \Delta Z_1 = -[\ \Delta Y_1^HY_1 + Y_1^H\Delta Y_1 + \bar{\Lambda}_1\Delta Y_1^HY_1\bar{\Lambda}_1' + \bar{\Lambda}_1Y_1^H\Delta Y_1\bar{\Lambda}_1']. \tag{8}$$

$$and, \Delta Z_2 = -[\ \Delta Y_1^H Y_1 + Y_1^H \Delta Y_1 \\ +\Lambda_1' \Delta Y_1^H Y_1 \Lambda_1 + \Lambda_1' Y_1^H \Delta Y_1 \Lambda_1]. \tag{9}$$

Substituting equations (9) and (8) in equation (7) we get: $\Delta J_1 = -2\ tr[Y_1\ (Z_1^2 Z_2 + Z_2^2 Z_1 \Lambda_1 Z_1^2 Z_2 \Lambda_1' + \bar{\Lambda}_1' Z_2^2 Z_1 \bar{\Lambda}_1) \Delta Y_1^H + (Z_1^2 Z_2 + Z_2^2 Z_1 + \Lambda_1 Z_1^2 Z_2 \Lambda_1' + \bar{\Lambda}_1' Z_2^2 Z_1 \bar{\Lambda}_1) Y_1^H \Delta Y_1]$

$$\therefore \Delta J_1 = -\ 2tr[Y_1 Z_3 \Delta Y_1^H + Z_3 Y_1^H \Delta Y_1] \\ = -\ 2tr[Y_1 Z_3 \Delta Y_1^H] - 2tr[Z_3 Y_1^H \Delta Y_1]. \tag{10}$$

We will now show that each of the terms $tr[Z_3 Y_1^H \Delta Y_1]$ and $tr[Y_1 Z_3 \Delta Y_1^H]$ can be expressed in terms of the quantities $U_1, \Gamma_1, X_1\ and\ B$.
First consider $tr[Z_3 Y_1^H \Delta Y_1]$.
$\Delta Y_1 = -K\bar{X}_1 \Delta \Psi_1$.

$$\text{Also, } Z = (MX_1\Lambda_1)^T Y_1 \Lambda_1' + (KX_1)^T Y_1 \\ = P^T Y_1 \Lambda_1' + Q^T Y_1 \\ = P^T \bar{P} \Psi_1 + Q^T \bar{Q}\ \Psi_1$$

$$\text{Thus, } Z = C_1\ \Psi_1. \tag{11}$$

$$\text{Hence, } \Delta\Psi_1 = C_1^{-1} \Delta Z \tag{12}$$

$$Thus, \Delta Y_1 = -K\bar{X}_1 C_1^{-1} \Delta Z \tag{13}$$

So, $tr[Z_3 Y_1^H \Delta Y_1] = -tr[Z_3 Y_1^H K \bar{X}_1 C_1^{-1} \Delta Z]$
Now since Z satisfies the Sylvester equation $\Lambda_1\ \mathbb{Z} - \mathbb{Z}\ \Lambda_1' = -\ \Lambda_1 X_1^T B \Gamma$,
we have,

$$\Lambda_1\ (\Delta Z) - (\Delta Z)\ \Lambda_1' = -\ \Lambda_1 X_1^T B (\Delta\Gamma). \tag{14}$$

Also from the analytical solution of a Sylvester equation [14] we can write:

$$\Delta Z = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\ \gamma_{jk}\ (\Lambda_1)^j\ (\Lambda_1 X_1^T B\ \Delta\Gamma\ )\ (\Lambda_1')^k.$$

$$So, tr[Z_3 Y_1^H \Delta Y_1] \\ = -tr[Z_3 Y_1^H K \bar{X}_1 C_1^{-1} \Delta Z] \\ = -tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\ \gamma_{jk}\ Z_4 (\Lambda_1)^j\ Z_5\ (\Lambda_1')^k] \\ = -tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\ \gamma_{jk}\ (\Lambda_1')^k Z_4 (\Lambda_1)^j\ Z_5]$$

Where, $Z_4 = Z_3 Y_1^H K \bar{X}_1 C_1^{-1}$ and $Z_5 = \Lambda_1 X_1^T B\ \Delta\Gamma$.
Since U_1 satisfy the Sylvester equation
$\Lambda_1'\ U_1 -\ U_1\ \Lambda_1 = -\ Z_3 Y_1^H K \bar{X}_1 C_1^{-1}$.
We can write

$$U_1 = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1}\ \gamma_{jk}\ (\Lambda_1')^k Z_4 (\Lambda_1)^j$$

So, finally we have;

$$tr[Z_3 Y_1^H \Delta Y_1] = tr[U_1 \Lambda_1 X_1^T B \Delta\Gamma] \tag{15}$$

Next, consider $tr[Y_1 Z_3 \Delta Y_1^H]$.

$$tr[Y_1 Z_3 \Delta Y_1^H] \\ = tr[Y_1 Z_3 (\Delta Y_1)^H] \\ = -tr[Y_1 Z_3 (\Delta Z)^H (C_1^{-1})^H X_1^T K]\quad (by 13) \\ = -tr[(C_1^{-1})^H X_1^T K Y_1 Z_3 (\Delta Z)^H]$$

$Since,\ \ \Lambda_1\ (\Delta Z) - (\Delta Z)\ \Lambda_1' = -\ Z_5,$
We have :
$\bar{\Lambda}_1' (\Delta Z)^H - (\Delta Z)^H \bar{\Lambda}_1 = (Z_5)^H$.
Also note that the solution $(\Delta Z)^H$ of the above Sylvester equation can be written as
$(\Delta Z)^H = \sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \delta_{jk} (\bar{\Lambda}_1')^j\ Z_5^H\ (\bar{\Lambda}_1)^k$.
Let $Z_6 = (C_1^{-1})^H X_1^T K Y_1 Z_3$.

$$Thus,\ tr[Y_1 Z_3 \Delta Y_1^H] \\ = -tr[Z_6 (\Delta Z)^H] \\ = -tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \delta_{jk}\ Z_6\ (\bar{\Lambda}_1')^j\ \{Z_5^H\}\ (\bar{\Lambda}_1)^k \\ = -tr[\sum_{j=0}^{p-1}\sum_{k=0}^{p-1} \delta_{jk} (\bar{\Lambda}_1)^k \{Z_6\} (\bar{\Lambda}_1')^j\ Z_5^H].$$

Now, define the matrix U_2 to be the unique solution of the Sylvester equation
$\bar{\Lambda}_1 \mathbb{U}_2 - \mathbb{U}_2 \bar{\Lambda}_1' = -Z_6$.

$$\text{Then, } tr[Y_1 Z_3 \Delta Y_1^H] \\ = -tr[U_2 (\Delta\Gamma)^H B^H \bar{X}_1 \bar{\Lambda}_1] \\ = -tr[B^H \bar{X}_1 \bar{\Lambda}_1 U_2 (\Delta\Gamma)^H] \tag{16}$$

Thus $\Delta J_1 = 2tr[U_1 \Lambda_1 X_1^T B \Delta\Gamma + B^H \bar{X}_1 \bar{\Lambda}_1 U_2 (\Delta\Gamma)^H]$

$$\therefore \bigtriangledown_\Gamma (J_1) = 2[U_1 \Lambda_1 X_1^T B]^T$$

The above results lead to the following algorithm:

Algorithm 1: *A Robust Partial Quadratic Eigenvalue Assignment Algorithm*
Inputs:

(i) The matrices M, D, and $K;\ M > 0,\ K = K^T,\ D = D^T$.
(ii) The control matrix B of order $n \times m\ (n \leq m)$.
(iii) A self conjugate set of complex numbers $\{\mu_1, \ldots, \mu_p\}$.
(iv) Tolerance $\in$, and the maximum number of iterations, Max_{iter}.

Output :
Real feedback matrices F and G such that the eigenvector matrix Y of the closed loop pencil has minimal condition number.
Step 0: Form the matrices $\Lambda_1,\ \Lambda_1',\ X_1,\ C_1$. Set k = 1.
Step 1: Choose a matrix $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\} \in C^{m\times p}$ such that if $\mu_j = \bar{\mu}_k$ then $\gamma_j = \bar{\gamma}_k$. Compute Y_1.
Step 2: Compute the solution U_1 of the Sylvester equation: $\Lambda_1'\ U_1 - U_1\ \Lambda_1 =\ Z_3 Y_1^H K \bar{X}_1 C_1^{-1}$. (This requires $(\frac{10}{3}+41)p^3$

flops.)
Step 3: Compute $Grad = \bigtriangledown_\Gamma(J_1)$. This requires $(n^2p + \frac{10}{3} + 41)p^3 + 17p^3 + 2p^2n$ flops
If $\|Grad\|_F < \epsilon$ or if the number of iterations exceed Max_{iter}, go to step 5. Else, Go to Step 4 .
Step 4: Compute a new Γ using a gradient based optimization method, set $k = k + 1$ and repeat from Step 2.
Step 5: Record the minimum value obtained for J_1 and corresponding value of Γ . For this Γ compute the matrices F and G, using formulas in Theorem 1. Stop.

Efficiency: The cost of the algorithm is dominated by solution of the Sylvester equation in Step 2 and the gradient evaluation of the Gradient in Step3. The flop-counts for these two steps are given above.It is to be noted that the matrices of the Sylvester equation in Step2 are diagonal matrices.
Computation of new Γ in Step 4:
The function to be minimized is $J_1 = \|(I_p - Y_1^H Y_1 - \Lambda_1' Y_1^H Y_1 \Lambda_1)^2\|_F^2$. Here Γ is a parameter and Y_1 is a function of Γ. We denote the current Γ by Γ_{old} and the new Γ by Γ_{new}. Then Γ_{new} can be obtained as follows:
i) Replace Γ_{old} by $\hat{\Gamma} = \Gamma_{old} + \alpha\ d_j$ where d_j is given by $d_j = -D_j\ Grad$. Here Grad represents the current gradient and D_j is the *metric* obtained as in the BFGS method.
ii) Obtain the value $\widehat{Y}_1$ of Y_1 corresponding to $\widehat{\Gamma}$, defined as follows:
Let $\widehat{Y}_1 = \{\widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_p\}$ and $\widehat{\Gamma} = \{\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_p\}$ then $\widehat{y}_i$ satisfies the equation $(M\mu_i^2 + D\mu_i + K)\ \widehat{y}_i = B\ \widehat{\gamma}_i$ for $i = 1 : p$.
iii) Find $l = \min_\alpha\ \|(I_p - \widehat{Y}_1^H \widehat{Y}_1 - \Lambda_1' \widehat{Y}_1^H \widehat{Y}_1 \Lambda_1)^2\|_F^2$. (This is done by using the MATLAB function *fminbnd*).
iv) $\Gamma_{new} = \Gamma_{old} + l\ d_j$.

VI. A Gradient Based Method for Simultaneous Improvement of Feedback Norms And Condition Number of Eigenvector Matrix

It is ideal to simultaneously improve both the feedback norms and the conditioning of the closed-loop eigenvalues. In this case, the objective function to be minimized can be represented as :

$$\mathbb{O} = \frac{[(C1)(\alpha)\mathbb{I} + (C2)(1-\alpha)\mathbb{J}_1]}{[(C1)(\alpha) + (C2)(1-\alpha)]} \quad (17)$$

where $\mathbb{I} = \frac{1}{2}[\|S\|_F^2 = \frac{1}{2}\ [\|F\|_F^2 + \ \ \|G\|_F^2]$ and $J_1 = \|(I_p - Y_1^H Y_1 - \Lambda_1' Y_1^H Y_1 \Lambda_1)^2\|_F^2$. and C1, C2 and α are constants. *Note that, when $\alpha = 1$, we have the minimum-norm problem and when $\alpha = 0$, we have the robust problem.*
The gradient of $\mathbb{O}$ with respect to Γ can be computed as

$$\bigtriangledown_\Gamma \mathbb{O} = \frac{[(C1)(\alpha) \bigtriangledown_\Gamma (\mathbb{I}) + (C2)(1-\alpha) \bigtriangledown_\Gamma (\mathbb{J}_1)]}{[(C1)(\alpha) + (C2)(1-\alpha)]}$$

$\bigtriangledown_\Gamma(\mathbb{I})$ is given by Theorem 2 and $\bigtriangledown_\Gamma(\mathbb{J}_1)$ is given by Theorem 5. If the magnitudes of the elements of $\bigtriangledown_\Gamma(\mathbb{I})$ and $\bigtriangledown_\Gamma(\mathbb{J}_1)$ are widely disparate then the constants C1 and C2 may be chosen to scale their magnitudes. Thus the problem of one gradient totally dominating the other in the calculation of the gradient of $\mathbb{O}$ can be avoided. After the constants C1 and C2 have been chosen appropriately for a particular problem, the value of the constant α is varied between 0 and 1, to bring about different amounts of reduction in the feedback norms and the condition number of the eigenvector matrix.

Numerical Examples:
The algorithms were tested on two test problems taken from [17], [2]. In the following, the percentage reduction in the Frobenius norm is calculated as **Percentage Reduction** $= 100 * \dfrac{\text{IN} - \text{FN}}{\text{IN}}$ where IN $=$ Value of the norm with initial Γ and FN $=$ Value of the norm with the finalΓ

Similarly, for the condition number.

Accuracy= The Frobenius norm of the difference between the desired and actual closed-loop eigenvalues.

Problem 1:
$\mathbf{M} = 4 * \mathbf{I_{n\times n}}$, $\mathbf{D} = 4 * \mathbf{I_{n\times n}}$

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

With $n = 10$, there are 20 open-loop eigenvalues of which the first six namely:
$\{-0.5 \pm 0.7375i, -0.5 \pm 0.8518i, -0.5 \pm 0.8090i\}$.
are reassigned to :
$\{-8 \pm .7375i, -8 \pm 0.8518i, -8 \pm 0.8090\}$, keeping other eigenvalues unchanged.

Problem 2:

$$\mathbf{K} = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{M} = 10I_{3,3}, \mathbf{D} = O_{3,3}$$

The open-loop eigenvalues are:
$\{\pm 3.6039i, \pm 2.49399i, \pm 0.8901i\}$.
The first two eigenvalues $\{\pm 3.6039i\}$ were reassigned to $\{-1, -2\}$, the other eigenvalues were kept unchanged.
The results of our numerical experiments are tabulated in the followings.

Comparison Of Algorithm 2 With the Qian-Xu method

We now present the results obtained by using the Qian-Xu method for Problems 1 and 2.
Note this method is only for condition number reduction only.

NOTE: Problem 1 was also tested for larger problems of sizes ranging from $n = 200$ to $n = 900$ with our

TABLE I

NUMERICAL RESULTS FOR PROBLEM 1

Method	Accuracy	Percentage Reduction
Norm Reduction	$0(10^{-4})$	99.93
Condition Number Reduction **(Algorithm 1)**	$0(10^{-6})$	97.85
Simultaneous Reduction of Norms and Condition Numbers $\alpha = 0.2$, $C1 = 0.5, C2 = 10^{16}$	$0(10^{-5})$	99.99

TABLE II

NUMERICAL RESULTS FOR PROBLEM 2

Method	Accuracy	Percentage Reduction
Norm Reduction	$0(10^{-14})$	99.75
Condition Number Reduction	$0(10^{-15})$	99.54
Simultaneous Reduction of Condition Number and Feedback Norms. $\alpha = 0.2$, $C1 = 10^6, C2 = 0.5$	$0(10^{-15})$	100

TABLE III

RESULTS OF THE QIAN-XU METHOD

Prob.	Percentage Reduction	Accuracy
1	98.56	$5.5108e - 007$
2	99.45	$2.1915e - 014$

algorithms.The results are similar to those presented here and appear in [4].

CONCLUSION

The problem of designing a robust active controller for a vibrating structure modeled by a system of second-order matrix differential equations is the one in which a feedback controller has to be constructed in such way that the feedback matrices have minimum norms and the closed-loop eigenvalues are as insensitive as possible to small perturbations of the data. Mathematically, this leads to minimum-norm and robust partial quadratic eigenvalue assignment problems (MNPQEVAP and RPQEVAP). Basically the problems are optimization problems and one special advantage of solving these problems in an optimization setting is that some of the excellent existing numerical optimization techniques can be profitably used. However, a bottleneck of using such techniques is to derive a parametric expressions for feedback matrices and develop appropriate gradient formulas. In case of the problems under consideration here a further computational challenge is to develop such gradient formulas using only a few eigenvalues and eigenvectors of the associated quadratic eigenvalue problem, since it is impossible to compute in practice the entire spectrum and the eigenvectors of a large quadratic matrix pencil even with the state-of-the-art computational techniques. In the present paper and in another recent one [5], meeting these challenges, (i) parametric expressions for feedback matrices via Sylvester equations have been derived, and (ii) appropriate gradient formulas both for both minimum-norm and robust eigenvalue assignment problems have been developed in terms of only a small number of eigenvalues that need to be reassigned and the associated eigenvectors, without reducing the order of the model.These techniques are, therefore, implementable in practice even for large-scale structures. However, some more work still needs to be done. One of the underlying mathematical problems is how to choose the initial parametric matrix in each such algorithm so that convergence can be guaranteed. Some more research is in order.

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