

ZAMM · Z. angew. Math. Mech. **00** (2000) 0, 1–12

DATTA, B. N.; RAM, Y. M.; SARKISSIAN, D. R.

Spectrum modification for gyroscopic systems

The problem of modifying the spectrum of gyroscopic systems by applying external forces is considered. The analysis of the problem leads to a certain inverse integro-differential eigenvalue problem in the distributed parameter case, and to a non-symmetric restricted rank modification in the finite-dimensional case. The explicit solution presented may be applied to stabilize structures and systems, where a small part of the spectrum is required to be assigned and the rest of the spectrum is to remain unchanged. The required forces are determined by using a partial knowledge of the eigenvalues which are intended to be changed, and their associated eigenfunctions (eigenvectors).

Keywords: Eigenvalue assignment, distributed parameter, gyroscopic

MSC (1991): 00X00, 00X00

1 Introduction

Consider an n degree-of-freedom vibratory system, attached to a rigid frame. Suppose that the frame rotates with a constant angular velocity. Then the free infinitesimal oscillations of the system about the frame are governed by the system of ordinary differential equations

$$M\ddot{u} + G\dot{u} + Ku = 0, \quad M, G, K \in \mathbb{R}^{n \times n}, \quad u \in \mathbb{R}^n, \quad (1)$$

where M and K are symmetric positive definite *mass* and *stiffness* matrices and $G = -G^T$ is a skew-symmetric *gyroscopic* matrix. The eigenvalues of the system (1) are the roots of the characteristic equation

$$\det(\lambda^2 M + \lambda G + K) = 0. \quad (2)$$

It is well known (see [5]) that with the conditions that we have imposed on M , G and K , the system (1) has $2n$ semi-simple purely imaginary eigenvalues (including multiplicity).

If a harmonic force $r \sin(\alpha t)$, r constant vector and α real scalar, is applied to the system (1) then the dynamics of the system is governed by

$$M\ddot{u} + G\dot{u} + Ku = r \sin(\alpha t). \quad (3)$$

The system (3) oscillates with large amplitude vibrations whenever $i\alpha$ is in the neighborhood of an eigenvalue λ_j . If $\lambda_j = i\alpha$ then the system *resonates*, its amplitude of oscillation increases with time without bound. To avoid resonance, or near resonance phenomena, the dynamics of the system may be altered by applying an external force of the form

$$(f^T \dot{u} + g^T u)b, \quad b, f, g \in \mathbb{R}^n. \quad (4)$$

The equations of free motion for the modified system are thus

$$M\ddot{u} + G\dot{u} + Ku = (f^T \dot{u} + g^T u)b, \quad (5)$$

or equivalently

$$M\ddot{u} + (G - bf^T)\dot{u} + (K - bg^T)u = 0. \quad (6)$$

Let

$$\Omega = \{ \{ \mu_i \}_{i=1}^m, \{ \lambda_j \}_{j=m+1}^{2n} \} \quad (7)$$

be a self conjugate set, where $\{ \mu_i \}_{i=1}^m$ is a prescribed set, closed under conjugation, and λ_j , $j = m+1, \dots, 2n$, are eigenvalues of (1). The problem under consideration is to select the vectors f and g such that the eigenvalues of the modified system, defined as the roots of the characteristic polynomial

$$\det(\lambda^2 M + \lambda(G - bf^T) + (K - bg^T)) = 0, \quad (8)$$

form precisely the set Ω .

Next consider the small oscillations of a taut string, rotating about its axis x with constant angular velocity ω_1 . It will be shown in §5 that the motion of the string is governed by the system of partial differential equations

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ \xi_{tt} \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} -\omega_1^2 - c^2(x) \frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\omega_1^2 - c^2(x) \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (9)$$

with vanishing boundary conditions at $x = 0$ and $x = 1$, where $\psi(x, t)$ and $\xi(x, t)$ are the displacements of an element of the string in the y - and z -direction, respectively, $c^2(x) > 0$ for all $0 \leq x \leq 1$, and the subscript t denotes a partial derivative with respect to time. Let

$$b = \begin{bmatrix} b_1(x) \\ b_2(x) \end{bmatrix}, \quad f = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \quad g = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} \quad \text{and} \quad u(t, x) = \begin{bmatrix} \psi(t, x) \\ \xi(t, x) \end{bmatrix}.$$

By application of an external force of the form

$$b \int_{x=0}^1 (f^T u_t + g^T u) dx, \quad (10)$$

we assign the eigenvalues of (9) to the set

$$\Omega = \{ \{ \mu_i \}_{i=1}^m, \{ \lambda_j \}_{j=m+1}^\infty \} \quad (11)$$

which leads to a spectral modification problem similar to that described by (6). where here, in the distributed parameter case, $n \rightarrow \infty$.

We now consider the small oscillations of a uniform string traveling with constant velocity γ over two fixed supports at $x = 0$ and $x = 1$. The motion of the moving string is governed by the partial differential equation

$$u_{tt} + 2\gamma u_{xt} + (\gamma^2 - c^2) v_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \quad \gamma^2 < c^2, \quad (12)$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad (13)$$

see e.g. [9]. With $f = f(x)$ and $g = g(x)$, we look for an external force of the form

$$b \int_{x=0}^1 (f u_t + g u) dx, \quad (14)$$

which assigns the eigenvalues of (12), (13) to the set Ω .

It is shown in §2 that the three systems described above belong to a certain class, called *gyroscopic* systems. A unified approach can thus be applied in selecting the required external forces to modify the spectra of the above mentioned systems.

Some spectral properties of gyroscopic systems are developed in §3, and an explicit solution to the partial assignment of the spectrum is presented in §4. Examples are given in §5 and conclusions are drawn in the last section.

2 Gyroscopic Systems

Denote the domain of the spatial coordinates by \mathbb{X} , and the relevant Hilbert space by $\mathbb{V} \subset \{f(x) : \mathbb{X} \rightarrow \mathbb{C}\}$. The scalar product (\cdot, \cdot) is such that $(\alpha v, \beta w) = (\beta w, \alpha v) = \bar{\alpha} \beta (v, w)$ for all $v, w \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{C}$, where bar denotes complex conjugation. Homogeneous boundary conditions are taken into account by restricting the displacement function $u(t, x) \in \mathbf{C}^2(\mathbb{R}, \mathbb{V})$, i.e. $u(t, x)$ is twice continuously differentiable function from the real numbers to the Hilbert space \mathbb{V} .

Let the dual of \mathbb{V} be denoted by \mathbb{V}' , and let a linear operator $A \in \mathbb{V}'$ be such that

$$(Av, w) = (v, Aw), \quad (Av, v) > 0, \quad (15)$$

for all $v, w \in \mathbb{V}$, $\|v\| \neq 0$. Then A is said to be a *self-adjoint positive definite* operator.

A linear operator $B \in \mathbb{V}'$ which satisfies

$$(Bv, w) = -(v, Bw), \quad (16)$$

for all $v, w \in \mathbb{V}$, is called a *gyroscopic* operator.

We now show that the three systems (1), (9) and (12) described in §1 fit into the general framework of

$$Mu_{tt} + Gu_t + Ku = 0, \quad (17)$$

where M and K are self-adjoint positive definite operators and G is gyroscopic.

Clearly with $\mathbb{X} = \{1, 2, \dots, n\}$, $\mathbb{V} = \mathbb{C}^n$ and the scalar product $(v, w) = \bar{v}^T w$, the system (1) falls into this framework.

Then with

$$\mathbb{X} = \{1, 2\} \times [0, 1],$$

$$\mathbb{V} = \{v = [v(1, x), v(2, x)]^T : v(i, x) \in \mathbf{C}^2([0, 1]); v(i, 0) = v(i, 1) = 0 \text{ for } i = 1, 2\},$$

and

$$(v, w) = \int_0^L \sum_{i=1}^2 \overline{v(i, x)} w(i, x) dx \quad \text{for any } v, w \in \mathbb{V},$$

we find that (9) can be written in the form (17) where

$$M = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -c^2(x) \frac{\partial^2}{\partial x^2} - \omega_1^2 & 0 \\ 0 & -c^2(x) \frac{\partial^2}{\partial x^2} - \omega_1^2 \end{bmatrix}.$$

For a sufficiently small ω the operator K is positive definite.

We now show that the system (12) is gyroscopic. Selecting

$$\mathbb{X} = [0, L], \quad \mathbb{V} = \{v(x) \in \mathbf{C}^2([0, 1]) : v(0) = v(1) = 0\},$$

$$(v, w) = \int_0^L \overline{v(x)} w(x) dx \quad \text{for any } v, w \in \mathbb{V},$$

$$Mv = v, \quad Gv = 2\gamma \frac{\partial v}{\partial x}, \quad Kv = (\gamma^2 - c^2) \frac{\partial^2 v}{\partial x^2},$$

gives

$$(Mv, w) = \int_0^L \overline{w(x)} v(x) dx = \overline{(Mw, v)} = (v, Mw),$$

$$(Mv, v) = \int_0^L |v(x)|^2 dx \geq 0.$$

Integrating by parts yields

$$(Kv, w) = - \int_0^L \overline{(\gamma^2 - c^2)w''(x)} v(x) dx = (v, Kw),$$

and

$$(Kv, v) = \int_0^L (\gamma^2 - c^2) |v'(x)|^2 dx > 0,$$

in view of the boundary conditions (13). Since $v'(x)$ does not vanish identically, K is a self-adjoint positive definite operator. Another integration gives

$$(Gv, w) = - \int_0^L \overline{2\gamma w'(x)} v(x) dx = -(v, Gw),$$

and hence the traveling string is a gyroscopic system.

From now on M and K are considered to be self-adjoint positive definite operators and G denotes a gyroscopic operator.

3 Some Spectral Properties of Gyroscopic Systems

With separation of variables $u(t, x) = e^{\lambda t}v(x)$ the system (17) reduces to the eigenvalue problem

$$(\lambda^2 M + \lambda G + K)v(x) = 0 \text{ and } v(x) \in \mathbb{V}. \quad (18)$$

The scalars λ_j and the corresponding functions $v_j(x)$ which non-trivially solves (18) are called eigenvalues and eigenfunctions, respectively.

Following [4, 5] the spectral properties of gyroscopic systems are well understood. For the sake of completeness we redevelop here some results needed for the analysis.

Proposition 1. *The eigenvalues of (18) are purely imaginary.*

Proof: Let λ and $v(x)$ be an eigenpair of (18). The scalar product of (18) with $\lambda v(x) \in \mathbb{V}$ gives

$$-(\lambda v, \lambda G v) = (\lambda v, \lambda^2 M v) + (\lambda v, K v), \quad (19)$$

$$-(\lambda G v, \lambda v) = (\lambda^2 M v, \lambda v) + (K v, \lambda v), \quad (20)$$

and by (15) and (16) we have

$$(\lambda v, \lambda G v) = (\lambda^2 v, \lambda M v) + (v, \lambda K v). \quad (21)$$

Adding (19) and (21) gives

$$0 = (\lambda + \bar{\lambda})(\lambda v, \lambda M v) + (v, K v). \quad (22)$$

Since $(\lambda v, M \lambda v) > 0$ and $(v, K v) > 0$ we necessarily have $\lambda + \bar{\lambda} = 0$, *i.e.* λ is purely imaginary. \square

A first order realization of (18) is given by

$$\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} v \\ \lambda v \end{bmatrix} \lambda = \begin{bmatrix} 0 & K \\ -K & -G \end{bmatrix} \begin{bmatrix} v \\ \lambda v \end{bmatrix}. \quad (23)$$

Denote $\hat{\lambda} = i\lambda$ and $\phi + i\eta = [v, \lambda v]^T$. Then, since λ is purely imaginary the system (18) has the following *symmetric* realization

$$\hat{\lambda} \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \eta \\ \phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & -K & -G \\ 0 & -K & 0 & 0 \\ K & G & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \phi \end{bmatrix}, \quad (24)$$

where the left-hand-side matrix is *positive definite*. If $\left\{ \hat{\lambda}, \begin{bmatrix} \eta \\ \phi \end{bmatrix} \right\}$ is an eigenpair of (24) then $\left\{ \hat{\lambda}, \begin{bmatrix} \eta - \phi \\ \eta + \phi \end{bmatrix} \right\}$, $\left\{ -\hat{\lambda}, \begin{bmatrix} -\eta \\ \phi \end{bmatrix} \right\}$ and $\left\{ -\hat{\lambda}, \begin{bmatrix} \eta + \phi \\ \eta - \phi \end{bmatrix} \right\}$ are also the eigenpairs of (24). It follows therefore that the real eigenpairs of (24) determine the generally complex eigenpairs of (18) uniquely, and *vice versa*.

We now introduce the Hilbert space $\mathbb{W} = \mathbb{V} \times \mathbb{V}$ associated with (23), such that for each $[v_1, v_2]^T, [w_1, w_2]^T \in \mathbb{W}$ the scalar product

$$\langle [v_1, v_2]^T, [w_1, w_2]^T \rangle = (K v_1, w_1) + (M v_2, w_2). \quad (25)$$

For each eigenpair $\{\lambda_j, v_j\}$ of (18) we thus have the counterpart $\{\lambda_j, \hat{v}_j\}$ of the first order realization (23), where

$$\hat{v}_j = \begin{bmatrix} v_j \\ \lambda_j v_j \end{bmatrix}, \quad (26)$$

and conclude that that

Proposition 2. *The eigenfunctions (26) are complete in \mathbb{W} .*

In analogy to the well known biorthogonal relations for the symmetric generalized eigenvalue problem see, e.g. [2, p. 507] the following result holds for gyroscopic systems.

Theorem 3. *Let v_j, v_k be two eigenfunctions of (18) associated with two eigenvalues $\lambda_j \neq \lambda_k$. Then*

$$\langle \hat{\mathbf{v}}_k, \hat{\mathbf{v}}_k \rangle = (\lambda_k M v_k, \lambda_j v_j) + (K v_k, v_j) = 0. \quad (27)$$

Proof: Similar to (19)-(20) we write

$$-(\lambda_k v_k, \lambda_j G v_j) = (\lambda_k v_k, \lambda_j^2 M v_j) + (\lambda_k v_k, K v_j), \quad (28)$$

and

$$-(\lambda_k G v_k, \lambda_j v_j) = (\lambda_k^2 M v_k, \lambda_j v_j) + (K v_k, \lambda_j v_j). \quad (29)$$

Then

$$(\lambda_k v_k, \lambda_j G v_j) = (\lambda_k^2 v_k, \lambda_j M v_j) + (v_k, \lambda_j K v_j), \quad (30)$$

by virtue of (15) and (16). Adding (28) and (30) gives

$$0 = (\bar{\lambda}_k + \lambda_j) ((\lambda_k v_k, \lambda_j M v_j) + (v_k, K v_j)), \quad (31)$$

and it thus follow from Proposition 1 that (27) holds. \square

If λ_p is a semi-simple eigenvalue of multiplicity q then we may choose q linearly independent eigenfunctions such that (27) holds. This can be done, for example, by using Gram–Schmidt orthogonalization.

4 Modifying the Spectrum

We consider here the following problem

Problem 1. *Suppose $M, G, K, b(x)$ and a complex set $\{\mu_j\}_{j=1}^m$ are given. Find two functions, $f(x)$ and $g(x)$, such that each element of Ω , defined by (7), is an eigenvalue of the modified system*

$$M v_{tt} + G v_t + K v = (f, v_t) b + (g, v) b. \quad (32)$$

Separation of variables $v(t, x) = e^{\mu t} w(x)$ gives

$$(\mu^2 M + \mu G + K) w = (f, \mu w) b + (g, w) b \quad \text{with } w(x) \in \mathbb{V}. \quad (33)$$

The pair $\{\mu_j, w_j(x)\}$ which non-trivially solves (33) is called an eigenpair of the modified system (32).

Theorem 4. *Suppose that the eigenvalues $\lambda_1, \dots, \lambda_m$ of (17) are distinct, $\mu_j \notin \{\lambda_k\}_{k \geq 1}$, and $(b, v_j) \neq 0$ for $j = 1, \dots, m$. Then the solution to Problem 1 is given by*

$$f(x) = \sum_{j=1}^m \beta_j \lambda_j M v_j, \quad g(x) = \sum_{j=1}^m \beta_j K v_j, \quad (34)$$

where

$$\beta_j = \frac{1}{(b, \lambda_j v_j)} \frac{\prod_{i=1}^m (\bar{\mu}_i + \lambda_j)}{\prod_{\substack{i=1 \\ i \neq j}}^m (\lambda_j - \lambda_i)}. \quad (35)$$

Proof: We first show that with (34) each $\{\lambda_k, v_k\}$, for $k > m$, is an eigenpair of (33) when β_1, \dots, β_m are arbitrarily chosen. Indeed, substituting (34) in (33) gives

$$\begin{aligned} \lambda_k^2 M v_k + \lambda_k G v_k + K v_k - b \left(\sum_{j=1}^m \beta_j \lambda_j M v_j, \lambda_k v_k \right) - b \left(\sum_{j=1}^m \beta_j K v_j, v_k \right) \\ = 0 - b \sum_{j=1}^m \bar{\beta}_j ((\lambda_j M v_j, \lambda_k v_k) + (K v_j, v_k)) = 0 \end{aligned}$$

by virtue of the orthogonal relation (27).

Let $w_k(x) \in \mathbb{V}$ be the solution of

$$\mu_k^2 M w_k(x) + \mu_k G w_k(x) + K w_k(x) = b(x). \quad (36)$$

Such a solution exists by the Fredholm alternative, since μ_k is not in the spectrum of (17).

We now show that with $\beta_1, \beta_2, \dots, \beta_m$ chosen according to (35), each $\{\mu_k, w_k\}$ is an eigenpair of (33). Similar to (28) and (29) we obtain for $j = 1, 2, \dots, m$

$$-(\mu_k w_k, \lambda_j G v_j) = (\mu_k w_k, \lambda_j^2 M v_j) + (\mu_k w_k, K v_j) \quad (37)$$

and

$$-(\mu_k G w_k, \lambda_j v_j) = (\mu_k^2 M w_k, \lambda_j v_j) + (K w_k, \lambda_j v_j) - (b, \lambda_j v_j). \quad (38)$$

Adding (37) and (38) gives

$$(b, \lambda_j v_j) = (\bar{\mu}_k + \lambda_j) ((\mu_k M w_k, \lambda_j v_j) + (K w_k, v_j)) \quad (39)$$

by virtue of (15), (16) and Proposition 1. Substituting (34), (35) in (33) and using (39) yields

$$\begin{aligned} \mu_k^2 M w_k + \mu_k G w_k + K w_k - b(f, \mu_k w_k) - b(g, w_k) \\ = b - b \sum_{j=1}^m \bar{\beta}_j ((\lambda_j M v_j, \mu_k w_k) + (K v_j, w_k)) = b \left(1 - \sum_{j=1}^m \frac{(b, \lambda_j v_j)}{\bar{\mu}_k + \lambda_j} \beta_j \right) \end{aligned} \quad (40)$$

Note that $K > 0$ implies that $\lambda_j \neq 0$ for all j . Write $\zeta_j = (b, \lambda_j v_j) \beta_j$, then the right-hand-side of (40) vanishes if and only if

$$C \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_m \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad (41)$$

where

$$C = \begin{bmatrix} (\bar{\mu}_1 + \lambda_1)^{-1} & \cdots & (\bar{\mu}_1 + \lambda_m)^{-1} \\ \vdots & & \vdots \\ (\bar{\mu}_m + \lambda_1)^{-1} & \cdots & (\bar{\mu}_m + \lambda_m)^{-1} \end{bmatrix}. \quad (42)$$

An explicit formula for the determinant of C , attributed in [7, p. 345] to [1], is given by

$$\det(C) = \frac{\prod_{1 \leq j < k \leq m} (\lambda_j - \lambda_k) \prod_{1 \leq j < k \leq m} (\bar{\mu}_j - \bar{\mu}_k)}{\prod_{j,k=1}^m (\bar{\mu}_j + \lambda_k)}. \quad (43)$$

It thus follows that with the conditions imposed, C is invertible, and there exists a *unique* solution $\zeta_1, \zeta_2, \dots, \zeta_m$ to (41).

Moreover, using (43) and Cramer's rule one may find

$$\zeta_j = \frac{\prod_{i=1}^m (\bar{\mu}_i + \lambda_j)}{\prod_{\substack{i=1 \\ i \neq j}}^m (\lambda_j - \lambda_i)}. \quad \square$$

It follows from the explicit solution given by (34) and (35) that the functions $f(x)$, $g(x)$ are real whenever each of the sets $\{\mu_j\}_{j=1}^m$ and $\{\lambda_j\}_{j=1}^m$ is self-conjugate.

We now establish the uniqueness of the solution in the case where the eigenfunctions of the first order realization (23) are complete in \mathbb{W} .

Theorem 5. *Suppose the eigenfunctions of (23) are complete in \mathbb{W} , $\lambda_1, \dots, \lambda_m$ are distinct, $\{\mu_j\}_{j=1}^m \cap \{\lambda_k\}_{k \geq 1} = \emptyset$, and $(b, v_k) \neq 0$ for all $k \geq 1$. Then the solution to Problem 1 is unique.*

Proof: We may generally express a solution to Problem 1 in the form

$$\begin{bmatrix} K^{-1}g \\ M^{-1}f \end{bmatrix} = \sum_{k \geq 1} \beta_k \begin{bmatrix} v_k \\ \lambda_k v_k \end{bmatrix}$$

for some constants β_k , by virtue of the completeness of the eigenfunctions. It thus follows that

$$g = \sum_{k \geq 1} \beta_k K v_k, \quad f = \sum_{k \geq 1} \beta_k \lambda_k M v_k. \quad (44)$$

It will now be shown that $\beta_k = 0$ for all $k > m$. Let $\{\lambda_j, v_j\}$ and $\{\lambda_j, w_j\}$ be normalized eigenpairs of (18) and (33) respectively, in the sense that $\|v_j\| = \|w_j\| = 1$. Then by (18) and (33) we have

$$(w_j, \lambda_j^2 M v_j) + (w_j, \lambda_j G v_j) + (w_j, K v_j) = 0 \quad (45)$$

and

$$(\lambda_j^2 M w_j, v_j) + (\lambda_j G w_j, v_j) + (K w_j, v_j) = (b, v_j) ((f, \lambda_j w_j) + (g, w_j)). \quad (46)$$

Subtracting (45) from (46) gives

$$0 = (b, v_j) ((f, \lambda_j w_j) + (g, w_j)) \quad (47)$$

by virtue of (15)-(16) and Proposition 1. Since $(b, v_j) \neq 0$ it follows from (33) and (47) that

$$\lambda_j^2 M w_j + \lambda_j G w_j + K w_j = 0.$$

Hence $\{\lambda_j, w_j\}$ is also an eigenpair of (17). Suppose that v_j and w_j are linearly independent. Then $\tilde{w} = (b, v_j)w_j - (b, w_j)v_j$ is an eigenfunction of (17). But $(b, \tilde{w}) = 0$ contradicts our assumption that b is not orthogonal to the eigenfunctions of (17). It thus concluded that $v_j = w_j$.

Since $b(x)$ is not identically zero it follows from (18) and (33) that

$$(f, \lambda_j v_j) + (g, v_j) = 0 \quad \text{for all } j > m. \quad (48)$$

Substituting (44) in (48) yields

$$\begin{aligned} \sum_{k \geq 1} (\beta_k \lambda_k M v_k, \lambda_j v_j) + \sum_{k \geq 1} (\beta_k K v_k, v_j) \\ = \sum_{k \geq 1} \bar{\beta}_k ((\lambda_k M v_k, \lambda_j v_j) + (K v_k, v_j)) = \bar{\beta}_j ((\lambda_j M v_j, \lambda_j v_j) + (K v_j, v_j)) = 0 \end{aligned}$$

in view of (27). The positive definiteness of M and K implies that $\beta_j = 0$ for all $j > m$. Hence f and g have the form (34). The proof is completed by the uniqueness of $\beta_1, \beta_2, \dots, \beta_m$, established in the proof of Theorem 2. \square

By the completeness of $\{\hat{v}_j\}_{j \geq 1}$ we have

$$\hat{w}_k = \begin{bmatrix} w_k \\ \mu_k w_k \end{bmatrix} = \sum_{j \geq 1} d_{kj} \hat{v}_j, \quad k = 1, 2, \dots, m, \quad (49)$$

and by (27)

$$\langle \hat{w}_k, \hat{v}_j \rangle = d_{kj} \langle \hat{v}_j, \hat{v}_j \rangle.$$

Hence (39) can be written in the form

$$\langle \hat{w}_k, \hat{v}_j \rangle = \frac{(b, \lambda_j v_j)}{(\bar{\mu}_k + \lambda_j)}$$

and it thus follows that

$$d_{kj} = \frac{(b, \lambda_j v_j)}{(\bar{\mu}_k + \lambda_j) \langle \hat{v}_j, \hat{v}_j \rangle}$$

for all $1 \leq k, j \leq m$. Using (42) we obtain

$$\begin{bmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_p \end{bmatrix} = C^{-1} \begin{bmatrix} \frac{\langle \hat{v}_1, \hat{v}_1 \rangle}{(b, \lambda_1 v_1)} (\hat{w}_1 - \sum_{i > m} d_{1i} \hat{v}_i) \\ \vdots \\ \frac{\langle \hat{v}_m, \hat{v}_m \rangle}{(b, \lambda_m v_m)} (\hat{w}_m - \sum_{i > m} d_{mi} \hat{v}_i) \end{bmatrix}.$$

The eigenfunctions of Problem 1 with f and g as in (34)-(35) are thus complete in \mathbb{V} . Therefore, it is concluded that the spectrum of (33) with (34) and (35) is precisely the set Ω .

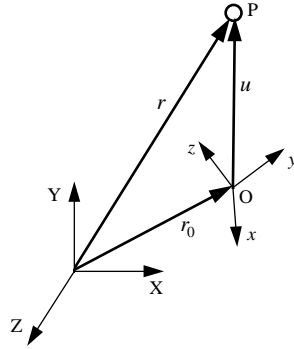


Figure 1: Coordinate systems

5 Modeling and Examples

Let X - Y - Z be a stationary coordinate system, and let x - y - z be another coordinate systems of origin O which moves in a general space motion with respect to the stationary coordinates, as shown in Figure 1.

Denote the angular velocity and angular acceleration of x - y - z by ω and $\dot{\omega}$. Let r and r_0 be the position vectors of a particle P and the origin of O of x - y - z , respectively. Denote the position of P with respect to the x - y - z coordinates by the vector u . Then in Newtonian dynamics we have

$$r = r_0 + u, \quad (50)$$

$$\dot{r} = \dot{r}_0 + \dot{u} + \omega \times u \quad (51)$$

and

$$\ddot{r} = \ddot{r}_0 + \ddot{u} + 2\omega \times \dot{u} + \omega \times (\omega \times u) + \dot{\omega} \times u, \quad (52)$$

where \dot{r} and \ddot{r} are the velocity and acceleration of P as observed from the stationary coordinates, \dot{u} and \ddot{u} are velocity and acceleration with respect to an observer fixed to x - y - z . Let R be the resultant of all forces applied to the particle P . Then by Newton's second law

$$R = m\ddot{u}, \quad (53)$$

where m is the mass of P .

Spatial oscillations of a particle. Let the particle of mass m be connected to a ring of radius α via two springs of constant κ and free length β . Suppose that the ring is rotating with constant angular velocity $\omega = (\omega_1, \omega_2, \omega_3)^T$. Let the moving frame x - y - z be attached to the ring, with its origin O at the center of the ring, and with its axis x aligned with \overline{PQ} , as shown in Figure 2. Then for small oscillations about the equilibrium position, Hook's law gives

$$R = \begin{bmatrix} -2\kappa & & \\ & -2\kappa\gamma & \\ & & -2\kappa\gamma \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

where $\gamma = (1 - \beta/\alpha)$, and u_1, u_2, u_3 are the displacements of the mass in the x -, y -, z -direction. Since O is a fixed point of rotation, $\ddot{r}_0 = 0$, and we have by (52)

$$\begin{aligned} \ddot{r} = \ddot{u} + 2\omega \times \dot{u} + \omega \times (\omega \times u) &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & -2\omega_3 & 2\omega_2 \\ 2\omega_3 & 0 & -2\omega_1 \\ -2\omega_2 & 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} -\omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_1^2 - \omega_3^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \end{aligned}$$

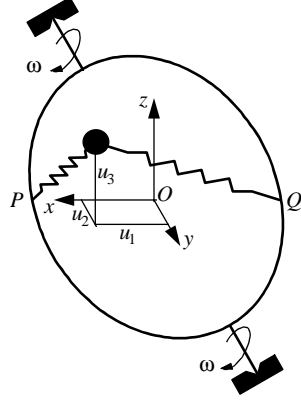


Figure 2: Spatial oscillations of a particle

By Newton's second law we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_3 & 2\omega_2 \\ 2\omega_3 & 0 & -2\omega_1 \\ -2\omega_2 & 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} 2\omega_n^2 - \omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & 2\omega_n^2\gamma - \omega_1^2 - \omega_1^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & 2\omega_n^2\gamma - \omega_2^2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (54)$$

where $\omega_n^2 = \kappa/m$. For sufficiently small ω the stiffness matrix K is positive definite.

For the sake of demonstration we set $\omega = (1, 2, 1)^T$, $\omega_n = 3$, $\gamma = 1/2$ and $b = (1, 0, 0)^T$. Each of

$$\left\{ 6.0860i, \begin{bmatrix} 0.0000 - 0.1171i \\ 0.0476 + 0.0318i \\ -0.0951 + 0.0159i \end{bmatrix} \right\}, \left\{ 3.1895i, \begin{bmatrix} 0.0000 - 0.0930i \\ -0.0277 - 0.2482i \\ 0.0554 - 0.1241i \end{bmatrix} \right\}, \\ \left\{ 0.8878i, \begin{bmatrix} 0.0000 + 0.2585i \\ 0.2945 - 0.2167i \\ -0.5890 - 0.1084i \end{bmatrix} \right\} \quad (55)$$

and their complex conjugates, is an eigenpair of (54).

We wish to assign the complex conjugate pair $\{0.8878i, -0.8878i\}$ to $\{-1 + i, -1 - i\}$. Using (34) and (35) we find

$$f = \begin{bmatrix} -2.0000 \\ 0.1217 \\ 3.9485 \end{bmatrix}, \quad g = \begin{bmatrix} -16.7625 \\ 14.9362 \\ -11.7818 \end{bmatrix}. \quad (56)$$

Small oscillations of a taut rotating string. Consider the taut string of density $\rho(x)$, stretched with tension T , attached to a rotating frame, as shown in Figure 3. Let the angular velocity of the frame be $\omega = (\omega_1, 0, 0)^T$. For an infinitesimal element of the string vibrating in the $y-z$ plane the Newton's second law gives

$$R = \begin{bmatrix} T\psi_{xx} \\ T\xi_{xx} \end{bmatrix}$$

where ψ and ξ are the displacements of the element of the string in the y - and z -direction, respectively. Since the origin of $x-y-z$ is stationary, (52) gives

$$\ddot{r} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ \xi_{tt} \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_1^2 \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix}.$$

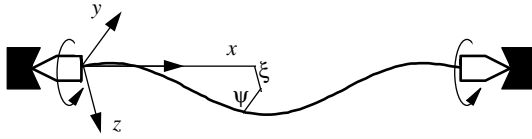


Figure 3: Small oscillations of a taut rotating string

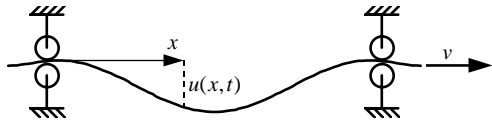


Figure 4: Small oscillations of traveling string

Hence by Newton's second law

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \psi_{tt} \\ \xi_{tt} \end{bmatrix} + \begin{bmatrix} 0 & -2\omega_1 \\ 2\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \psi_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} -\omega_x^2 - c^2(x) \frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\omega_x^2 - c^2(x) \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} \psi \\ \xi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $c^2(x) = T/\rho(x)$. The system has the following eigenpairs

$$\left\{ \lambda_k, e^{\pm \frac{\lambda_k - i}{2} x} \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}, \left\{ -\lambda_k, e^{\pm \frac{i - \lambda_k}{2} x} \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}, \tag{57}$$

where $\lambda_k = i(1 + 2\pi k)$ for all integers $k \neq 0$. Note that the eigenvalues $\pm\lambda_k$ have multiplicity two. However, for each double eigenvalue there exist two linearly independent eigenfunctions. The problem is thus *non-defective*.

To demonstrate the partial spectral modification we chose $\omega_1 = 1$ and $c^2 \equiv 4$. We wish to assign the complex conjugate pair $\{i(1 + 2\pi), -i(1 + 2\pi)\}$ to $\{-1 + i, -1 - i\}$ using $b(x) \equiv (1, 1)^T$. Following (34) and (35) we obtain

$$f = \frac{\pi}{4 + 8\pi} \begin{bmatrix} (3 - 4\pi^2) \cos(\pi x) + (1 + 8\pi + 4\pi^2) \sin(\pi x) \\ (1 + 8\pi + 4\pi^2) \cos(\pi x) - (3 - 4\pi^2) \sin(\pi x) \end{bmatrix} \tag{58}$$

$$g = \frac{\pi - 2\pi^2}{4 + 8\pi} \begin{bmatrix} (1 + 8\pi + 4\pi^2) \cos(\pi x) - (3 - 4\pi^2) \sin(\pi x) \\ (4\pi^2 - 3) \cos(\pi x) - (1 + 8\pi + 4\pi^2) \sin(\pi x) \end{bmatrix}. \tag{59}$$

It may be confirmed by direct substitution that despite the applied force, all other eigenpairs remain unchanged.

Small oscillations of a traveling string. With $c = 1$, $\gamma = 0.5$ and $b \equiv 1$ the eigenpairs of the traveling string, modeled by (12) and (13), are

$$\left\{ \lambda_k, e^{-\frac{2}{3}\lambda_k x} - e^{2\lambda_k x} \right\}, \tag{60}$$

where $\lambda_k = \frac{3}{4}\pi i(\pm 1 + 4k)$ for all integers k . The functions

$$f(x) = -\frac{1}{32} \left((9\pi^2 + 24\pi - 32) \cos\left(\frac{\pi x}{2}\right) + (9\pi^2 - 24\pi - 32) \sin\left(\frac{\pi x}{2}\right) \right) \sin(\pi x) \tag{61}$$

$$g(x) = \frac{\pi}{256} \left(9(9\pi^2 + 24\pi - 32) \cos\left(\frac{3\pi x}{2}\right) + (32 - 24\pi - 9\pi^2) \cos\left(\frac{\pi x}{2}\right) \right. \\ \left. + (9\pi^2 - 24\pi - 32) \left(\cos\left(\frac{\pi x}{2}\right) + 9 \sin\left(\frac{3\pi x}{2}\right) \right) \right) \tag{62}$$

assign the pair $\{\frac{3}{4}\pi i, -\frac{3}{4}\pi i\}$ to $\{-1 + i, -1 - i\}$. The modified system (33) has the eigenfunction

$$\frac{i}{2}e^{-(2+\frac{2}{3}i)x} \left(e^{(2+\frac{2}{3}i)x} - \frac{(1 - e^{-2+2i}) e^{\frac{8}{3}x}}{e^{\frac{2}{3}-\frac{2}{3}i} - e^{-2+2i}} - \frac{e^{2+\frac{8}{3}ix}}{e^{2i} + e^{\frac{2}{3}+\frac{4}{3}i} + e^{\frac{4}{3}+\frac{2}{3}i} + e^2} \right)$$

and its complex conjugate, corresponding to the eigenvalues $-1 + i$ and $-1 - i$, respectively. All other eigenpairs are unaltered by the applied force.

6 Concluding Remarks

We now comment on the possible practical significance of the presented approach. Reduction of vibrations in the traveling string problem has a considerable engineering importance. In [11] we read:

... the axially moving string equation is controllable and observable by a single point actuator and sensor which satisfy

$$\sin(k\pi x_a) \neq 0, \sin(k\pi x_s) \neq 0 \text{ with } x_a, x_s \in (0,1) \text{ for } k = 1, 2, \dots$$

where x_a and x_s are the locations of the actuator and the sensor, respectively. Satisfaction of these conditions requires x_s and x_a to be irrational.

Since the set of non-admissible points is dense, some difficulties may arise in application of the traditional optimal control approach. Moreover, use of any finite number of point actuators will not circumvent this difficulty.

With our approach, however, we may change in a desired manner the dynamics of the string (in a weak solution sense) by using a single point actuator. It should be noted, however, that the entire state must be accessible.

We do not require *full* controllability of the system, that is

$$(b, v_j) = v_j(x_a) \neq 0 \text{ for } j = 1, 2, \dots,$$

where v_j are given by (60). Only the *partial controllability* with respect to the eigenpairs which are intended to be changed, that is

$$(b, v_j) = v_j(x_a) \neq 0 \text{ for } j = 1, 2, \dots, m,$$

is needed.

In the present state of the art there is no systematic way to model damping of vibratory systems and structures. One popular model is the linear viscous damping force given by Du_t , where D is non-negative definite self-adjoint operator. It has been shown in [10] that such a model can be reconstructed from spectral data. We note in passing that if G is replaced by D in (18), then a similar force may be used to assign part of the spectrum. All that is needed to be done is to replace (34) and (35) with

$$f = \sum_{j=1}^m \beta_j \lambda_j M v_j, \quad g = - \sum_{j=1}^m \beta_j K v_j, \quad \beta_j = \frac{1}{(b, \lambda_j v_j)} \frac{\prod_{i=1}^m (\mu_i - \lambda_j)}{\prod_{i=1, i \neq j}^m (\lambda_i - \lambda_j)}.$$

This result has been established for the finite dimensional case in [3].

References

- 1 CAUCHY: Mémoire sur les fonctions alternées et sur les sommes alternées. Exercices d'analyse et de phys. math., ii (1841), 151–159; or Œuvres complètes, 2^e, sér. xii.
- 2 DATTA, B. N.: Numerical Linear Algebra and Applications. Brooks/Cole publishing Co., 1995.
- 3 DATTA, B. N.; ELHAY, S.; RAM, Y. M.: Orthogonality and Partial Pole Assignment for the Symmetric Definite Quadratic Pencil. Linear Algebra and its Applications, **257** (1997), 29–48.
- 4 DUFFIN, R. J.: A minimax theory for overdamped networks. Jour. Rat. Mech. and Anal., **4** (1955), 221–233.
- 5 DUFFIN, R. J.: The Rayleigh-Ritz method for dissipative or gyroscopic systems. Quarterly of Applied Mathematics, **18** (1960), 215–221.
- 6 INMAN, D. J.: Vibration: with control, measurement, and stability. Prentice Hall: Englewood Cliffs, N. J., 1989.

- 7 MUIR, T., SIR: The theory of determinants in the historical order of development, volume 1. Dover Publication, N. Y., 1960.
- 8 RAM, Y. M.: Pole assignment for the vibrating rod. Quarterly Journal of Mechanics and Applied Mathematics, **51** (1998) 3, 461–476.
- 9 RAM, Y. M.; CALDWELL, J.: The free vibrations of an axially moving string in a bounded region. Canadian Applied Mathematics Quarterly, **3** (1995) 4, 445–471.
- 10 RAM, Y. M.; ELHAY S.: An inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with applications to damped oscillatory systems. SIAM Journal of Applied Mathematics, **56** (1996), 232–244.
- 11 YANG, B.; MOTE, C. D., JR.: Active Vibration Control of the Axially Moving String in the S -Domain. Transactions of the ASME, Journal of Applied Mechanics, **58** (1991), 189–196.

Received August 0, 2000, revised August 0, 2000, final revision August 0, 2000, accepted August 0, 2000

Addresses: Prof. BISWA N. DATTA; DANIL R. SARKISSIAN, Northern Illinois University, Dept. of Mathematics, DeKalb, IL 60115, USA, email: dattab,sarkiss@math.niu.edu; Prof. YITSHAK M. RAM, Louisiana State University, Mechanical Engineering Dept., Baton Rouge, LA 70803, USA, email: ram@me.lsu.edu