

**RECENT ADVANCES  
ON  
COMPUTATIONAL METHODS  
FOR  
STRUCTURED INVERSE QUADRATIC  
EIGENVALUE PROBLEMS**

by

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# Two Inverse Quadratic Eigenvalue Problems

## I. Quadratic Partial Eigenvalue Assignment Problem (QPEVAP)

**Controlling Dangerous Vibrations  
in Structures**



**QPEVAP**

## II. Finite Element Model Updating Problem (**FEMUP**).

**Updating Theoretical FEM Using  
Measured Data from Real-Life  
Structure**



**FEMUP**

$\equiv$

**Structure preserving  
QPESAP**

## The Quadratic Eigenvalue Problem:

$$(\lambda^2 M + \lambda D + K)x = 0$$

- $2n$  eigenvalues and  $2n$  corresponding eigenvectors.
- The eigenvalues are the roots of the quadratic pencil  $\det(\lambda^2 M + \lambda D + K) = 0$ .

### • Quadratic Matrix Pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K$$

## Generalization of Standard Eigenvalue Problem

$$Ax = \lambda x$$

and  
the **Generalized Eigenvalue Problem**

$$Ax = \lambda Bx.$$

## Approach I

- Reduction to a Standard  $2n \times 2n$  Eigenvalue Problem

$$Au = \lambda u$$

where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

$$u = \begin{pmatrix} x \\ \lambda x \end{pmatrix}.$$

**(Assuming that  $M$  is nonsingular)**

- The eigenvalues are the same
- The eigenvectors are extracted from the eigenvectors  $u$ .

## Numerical Difficulties

- $M$  is **ill-conditioned**.
- Special structural properties: *definiteness*, *sparsity*, *bandness*, etc. **destroyed**.

- Reduction to a Generalized Eigenvalue Problem:

*Symmetric Generalized Eigenvalue Problem*

$$Bz = \lambda Cz$$

- $B = \begin{pmatrix} D & K \\ K & 0 \end{pmatrix}$

- $C = \begin{pmatrix} -M & 0 \\ 0 & K \end{pmatrix}$

- $z = \begin{pmatrix} \lambda x \\ x \end{pmatrix}$

## Numerical Difficulties

The pencil  $Bz = \lambda Cz$  is symmetric, but in general **indefinite**, even though  $M$ ,  $K$ , and  $D$  are symmetric positive definite.

**Remark:** The QEP is **nonlinear eigenvalue problem** - *difficult to solve*.

## State - of the - Art Methods.

- A **Look-ahead Lanczos Algorithm** of Parlett and Chen (1980) (only a few extremal eigenvalues).
- The **Jacobi-Davidson Method** (Projection Method).

*Only a few extremal eigenvalues and eigenvectors computed.*

## **Applications of the QEP.**

- Vibration Analysis of Structural Mechanical and Acoustic Systems
- Electrical Circuit Simulation
- Fluids Mechanics
- Modeling Microelectronic
- Finite-Element Model Updating in Aerospace and Automobile Industries.

## Quadratic Inverse Eigenvalue Problems.

- Certain inverse eigenvalue problems for the quadratic pencil arising in practical applications can be handled with a small number of eigenvalues and eigenvectors, if done properly.

## Examples of Resonance

Dangerous vibrations such as **resonance** are caused by a few bad eigenvalues.

### **Classical Examples of Resonance:**

- The Fall of the Tacoma Bridge
- The Fall of the Broughton Bridge in England
- Wobbling of the Millennium Bridge over the River Thames in London, England  
([www.arup.com/Millenniumbridge](http://www.arup.com/Millenniumbridge))

## Phenomenon of Resonance

- **The Discretized Finite Element Model**

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0.$$

- **The Associated Quadratic Matrix Eigenvalue Problem:**

$$(\lambda^2 M + \lambda D + K)x = 0.$$

- The dynamics are governed by

**Natural Frequencies**  $\longrightarrow$  Eigenvalues of the QEP.

**Mode Shapes**  $\equiv$  Eigenvectors of the QEP.

## Response of a Structure due to Harmonic Input

$$j = \sqrt{(-1)}.$$

- $f(t) = \text{External Force} = f_o e^{j\omega t}$
- Oscillatory Solution  $x(t) = x(t)e^{j\omega t}$
- $(K + j\omega D - \omega^2 M)x e^{j\omega t} = f_o e^{j\omega t}$
- $x = (K + j\omega D - \omega^2 M)^{-1} f_o$  (**Response**).

As

$$j\omega \rightarrow \lambda_j$$

$\|P(j\omega)^{-1}\|$  increases without bound.

- Resonance is caused by closed proximity of an external frequency to that of a natural frequency.

## How to Avoid Resonance?

- Feedback Control can be used

**Idea:** Replace {computed Unwanted eigenvalues}  
→ {suitably chosen ones}

and

Leave the remaining large number unchanged.

**(No spill-over)**

## Feedback Control in Second-order Model

**A possible Remedy:** Apply a suitable control force to the structure. Use the technique of **feedback control**.

- **Matrix Second-order Model with Control**

$$\boxed{M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t)}$$

$B$  - Control Matrix  
 $u(t)$  - Control Vector

- **Second-order Feedback Closed-loop System**

Choose  $u(t) = F_1\dot{x}(t) + F_2x(t)$ .

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = B(F_1\dot{x}(t) + F_2x(t))$$

$$\boxed{M\ddot{x}(t) + (D - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0.}$$

The associated matrix quadratic pencil:

- $P_c(\lambda) = \boxed{\lambda^2 M + \lambda(D - BF_1) + (K - BF_2) = 0.}$

This pencil is called the **closed-loop pencil**.

## Notations

- The spectrum of the quadratic pencil:

$$\Omega(P(\lambda)) = \{\lambda_1, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{2n}\}$$

- The right eigenvectors of the:

$$\{x_1, \dots, x_p; x_{p+1}, \dots, x_{2n}\}$$

- The left eigenvectors of the pencil:

$$\{y_1, \dots, y_p; y_{p+1}, \dots, y_{2n}\}.$$

## Quadratic Partial Eigenvalue Assignment Problem (QPEVAP)

### Given

- The system matrices  $M, K, D, \in \mathbb{R}^{n \times n}$  ( $M = M^T > 0$ ,  $K = K^T \geq 0$  and  $D = D^T$ ).
- A control matrix  $B \in \mathbb{R}^{n \times m}$

**Find** the Feedback Matrices  $F_1$  and  $F_2$  such that

$$\Omega(P_c(\lambda)) = \{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}\}.$$

- $\{\text{Unwanted Eigenvalues}\} \longrightarrow \{\text{User's Chosen Eigenvalues}\}$
- $\{\text{Good Eigenvalues}\} = \{\mathbf{Remain Unchanged}\}$

# Stabilizing a Second-order System

## (A Special Case)

- Solution of the QPEVA problem can be used to stabilize a matrix second-order system by feedback.

## Two Standard Approaches for Control

- Solution via transformation to a **first-order State-Space Form**
- **Independent Modal Space Control (IMSC)** Approach.

*Both these approaches have severe computational difficulties and engineering limitations.*

## Approach I

### Standard First-order Reduction

Recall the second-order feedback control system

$$M\ddot{x}(t) + (D - BF_1)\dot{x}(t) + (K - BF_2)x(t) = 0.$$

- Reduction to Standard First-order State-space Form:

$$\dot{q}(t) = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} q(t) + \begin{pmatrix} 0 \\ M^{-1}B \end{pmatrix} u(t)$$

### Opportunities

- Many numerically excellent methods can be used  
(**Numerical Methods for Linear Control Systems Design and Analysis**, by B.N. Datta, Elsevier Academic Press, 2003)

## Difficulties

- Ill-conditioned matrix inversion might be necessary.
- All important structures such as *sparsity*, *definiteness* and *bandness* etc. are lost.
- Problem size becomes double.

## Non-standard first-order reduction:

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \dot{z}(t) = \begin{pmatrix} 0 & -K \\ -K & -D \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ B \end{pmatrix} u(t)$$

or

$$E\dot{z}(t) = Az(t) + \hat{B}u(t) \text{ (Descriptor System)}$$

- Numerical methods for descriptor systems not well-developed ( $E$  could be *singular* or *very ill-conditioned*)
- Symmetry preserved, but *not Positive Definiteness, Sparsity and other Properties.*

**Approach II**  
**Independent Space Control (IMSC)**  
**Approach.**

**(For Open-loop Decoupling)**

- Requires complete knowledge of the spectrum and eigenvectors of the open-loop pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K.$$

**Impractical for large and sparse problems**

(For closed-loop Decoupling)

$$BKM^{-1}D = DM^{-1}BK$$

$$BKM^{-1}K = KM^{-1}BK$$

- Stringent requirements need to be satisfied on actuators and sensors which are impossible to satisfy in practice.

**Ref: Vibration with Control, Measurement, and Stability** by D. Inman, Prentice Hall, 1989.

## Challenges

- Use a **small number of eigenvalues and eigenvectors** that can be computed or measured.
- **No transformation** to a first-order system.
- **No reduction of the order** of the model or the order of the controllers.
- **Mathematical guarantee** needed for the **no spillover property**.

## The Current Engineering Practice and Drawbacks

- Compute and control the first few frequencies and mode shapes (eigenvalues and eigenvectors).
- Hope that the large number of remaining eigenvalues and eigenvectors do not change or do not spill-over to dangerous regions.
- **Unfortunately, the spill-over almost always occurs.**
- **No mathematical basis**

## Recent Direct and Partial-Modal Approach for Feedback Control

(Collaborative work with **Eric Chu, Sylvan Elhay, Yitshak Ram, Daniil Sarkissian, W.W. Lin, J.N. Wang, and others**)

- **Direct** - No transformation required.
- **Partial-Modal** - Only knowledge of a small number of eigenvalues and eigenvectors needed for implementation.
- Extension to the **Robust Partial Eigenvalue Assignment**. (Sensitivity minimization by minimization of the *eigenvector condition number and feedback normly*)

# A New Approach for the Quadratic Partial Eigenvalue Assignment Problem

- Two-part solution

**Part I. No spill-over part** (with a parametric matrix).

**Part II. Partial Eigenvalue Assignment Part.**  
(with a special choice of the parametric matrix)

## Notations

Define  $\Lambda_1 = \text{diag} (\lambda_1, \dots, \lambda_p)$

$Y_1 = (y_1, y_2, \dots, y_p)$

$\Lambda_{cl} = \text{diag} (\mu_1, \dots, \mu_p)$ .

## Solution of Part I

### Theorem on No Spill-over

- Choose any **arbitrary parametric matrix**  $\Phi$
- Define

$$F_1 = \Phi Y_1^H M$$

and

$$F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H D)$$

Then

$$\Omega(\lambda^2 M + \lambda(D - BF_1) + (K - BF_2)) = \{**\cdots*, \lambda_{p+1}, \dots, \lambda_{2n}\}.$$

**No Change.**

**Note:** Only small number of eigenvalues needed for constructing  $F_1$  and  $F_2$ .

# New Orthogonality Results on the Eigenvectors of the Quadratic Matrix Pencil

Assume

$$\{\lambda_1, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_{2n}\} = \phi.$$

Partition  $\Lambda = \text{diag} (\Lambda_1, \Lambda_2)$

$$X = (X_1, X_2)$$

$$Y = (Y_1, Y_2)$$

Then

$$\bullet \Lambda_1 Y_1^H M X_2 \Lambda_2 - Y_1^H K X_2 = 0$$

and

$$\bullet \Lambda_1 Y_1^H M X_2 + Y_1^H M X_2 \Lambda_2 + Y_1^H D X_2 = 0.$$

## Generalization of Orthogonality Results of SEVP and SDGEVP

- $X^T A X = \text{Diagonal}$  (Symmetric EVP)
- $\left. \begin{array}{l} X^T A X = \text{Diagonal} \\ X^T B X = I \end{array} \right\} \text{Symmetric Definite GEVP}$

## **Solution of Part II (How to Choose $\Phi$ ?)**

## Theorem on Partial Eigenvalue Assignment

- Solve the  $p \times p$  Sylvester Equation.

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H B F$$

- Solve the  $p \times p$  Linear system  $\Phi Z_1 = \Gamma$  by choosing  $\Gamma$  arbitrary.

- **Result:**

$$\Omega(\lambda^2 M + \lambda(D - B F_1) + (K - B F_2)) =$$

$\{\mu_1, \dots, \mu_p;$	$\lambda_{p+1}, \dots, \lambda_{2n}\}.$
<b>Desired EVS</b>	<b>No Change</b>

## An Algorithm for QPEVAP

**Step 1.** Form

- $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_p)$
- $Y_1 = (y_1, \dots, y_p)$
- $\Lambda_{c1} = \text{diag}(\mu_1, \dots, \mu_p)$ .

**Step 2.** Choose arbitrary  $m \times 1$  vectors  $\gamma_1, \dots, \gamma_p$  in such a way that  $\overline{\mu_j} = \mu_k$  implies  $\overline{\gamma_j} = \gamma_k$  and form

$$\Gamma = (\gamma_1, \dots, \gamma_p).$$

**Step 3.** Find the unique solution  $Z_1$  of the  $p \times p$  Sylvester equation

$$\Lambda_1 Z_1 - Z_1 \Lambda_{cl} = Y_1^H B \Gamma.$$

If  $Z_1$  is ill-conditioned, then return to Step 2 and select different  $\gamma_1, \dots, \gamma_p$ .

**Step 4.** Solve  $\Phi Z_1 = \Gamma$  for  $\Phi$ .

**Step 5.** Form  $F_1 = \Phi Y_1^H$  and  $F_2 = \Phi(\Lambda_1 Y_1^H M + Y_1^H D)$ .

- Standard Numerical Methods for Solving Sylvester and Lyapunov Equations
- Numerical Methods for Linear Control Systems  
(Chapter 8).

## Computing Resources and Requirements for Implementations

- A small number of eigenvalues and eigenvectors
- Solution of a small Sylvester equation
- Solution of a small linear algebraic system

## Practical and Computational Features

- Applicable to even very large real-life structures
- No transformation or model reduction
- Suitable for high-performance computing  
**(Rich in BLAS-3 Computations.)**
- Sparsity, bandness, symmetry, etc. can be exploited
- Mathematical guarantee of no spill-over
- Extension to more general problem of both partial **eigenvalue** and **eigenvector assignment (QPESA)**
- Generalization to the Partial Eigenvalue Assignment in **DPS. (Infinite Dimensions).**

## Quadratic Partial Eigenstructure Assignment Problem (QPEASP)

### Given

- The system matrices  $M, K, D, \in \mathbb{R}^{n \times n}$  ( $M = M^T > 0$ ,  $K = K^T \geq 0$  and  $D = D^T$ ).
- A set of computed unwanted eigenvalues  $\{\lambda_1, \dots, \lambda_p\}$ .
- A set of user's chosen eigenvalues  $\{\mu_1, \dots, \mu_p\}$ .
- A set of user's chosen eigenvectors  $\{x_{c1}, \dots, x_{cp}\}$

**Find** the Feedback Matrices  $F_1$  and  $F_2$  and a control matrix  $B$  such that

$$\Omega(P_c(\lambda)) = \{\mu_1, \dots, \mu_p; \lambda_{p+1}, \dots, \lambda_{2n}\}.$$

The Eigenvectors of  $p_c(\lambda) = \{x_{cl}, \dots, x_{cp}; x_{p+1}, x_{2n}\}$ .

{Unwanted Eigenvalues and Eigenvectors}  $\longrightarrow$  {User's Chosen Eigenvalues and Eigenvectors}

{Remaining Eigenvalues and Eigenvectors  $\longrightarrow$  No Change.}

## An Algorithm for QPESA

**Step 1.** Form  $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,

$$Y_1 = (y_1, \dots, y_p),$$

$$\Lambda_{c1} = \text{diag}(\mu_1, \dots, \mu_p), \quad \text{and } (X_{c1}, \dots, x_{cp}).$$

**Step 2.** Form the matrix

$$Z_1 = \Lambda_1 Y_1^H M X_{c1} + Y_1^H M X_{c1} \Lambda_{c1} + Y_1^H C X_{c1}.$$

Stop if  $Z_1$  is singular and conclude that the eigenstructure assignment with the given sets of eigenvalues and eigenvectors is not possible.

**Step 3.** Form the matrix  $T_c$  such that  $T_c \Lambda_{c1} T_c^H$  is a real matrix.

**Step 4.** Form

$$B = (M X_{c1} \Lambda_{c1}^2 + C X_{c1} \Lambda_{c1} + K X_{c1}) T_c^H,$$

$$F_1 = T_c Z_1^{-1} Y_1^H M, \text{ and}$$

$$F_2 = T_c Z_1^{-1} (\Lambda_1 Y_1^H M + Y_1^H C)$$

by solving the appropriate linear systems.

- There also exists a parametric Algorithm (as that of QPEVA)

(Ph.D Thesis by **Daniil Sarkissian**, Northern Illinois University, 2001).

**Natural Mathematical Model**

Distributed Parameter Systems



FEM



**Discretized Finite Element Model**

System of Second-order ODE.

- **Distributed Parameter Systems Model (DPS)**

### **Distributed Parameter Systems:**

$$M(x)\frac{\partial^2\nu(t, x)}{\partial t^2} + C(x)\frac{\partial\nu(t, x)}{\partial t} + K(x)\nu(t, x) = 0.$$

$M, C,$  and  $K$  are **differential operators** in the  $x$ -domain (spatial domain) of the displacement function  $\nu(t, x)$ .

$\nu(t, x)$  belongs to some Hilbert space.

$M =$  **Mass operator** (Self Adjoint)

$K =$  **Stiffness operator** (Self Adjoint)

$C = D + G$

$D =$  **Damping operator**

$G =$  **Gyroscopic operator** (Skew Symmetric)

DPS problems are **infinite dimensional**.

## **Two Additional Fundamental Challenges**

- Use finite dimensional control and computational techniques
- Guarantee the invariance of the finite spectrum mathematically.

# Mathematical Statement of the PEVA in DPS

## Given

- The operators  $M$ ,  $C$ , and  $K$ , of the DPS
- A self conjugate set of numbers  $\{\mu_1, \dots, \mu_p\}$
- Suitable control functions  $b_1, \dots, b_m$ .

**Find** Real Feedback Functions  $f_{11}, \dots, f_{1m}$  and  $f_{21}, \dots, f_{2m}$  such that

$$\begin{aligned} \Omega(P_c(\lambda)\phi) = & \lambda^2 M\phi + \lambda(C\phi - \sum_{k=1}^m (f_{1k}, \phi)_k) \\ & + (K\phi - \sum_{k=1}^m (f_{2k}, \phi)_k) \end{aligned} \quad (1)$$

is the set  $S = \{\mu_1, \dots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \dots\}$ .

### III. Partial Eigenvalue Assignment (PEVA) in Distributed Parameter Systems

Reassign a small part of the infinite open-loop spectrum of the operator pencil  $P(\lambda) = \lambda^2 M + \lambda C + K$ , by using feedback such that

- i. the set is replaced by a suitable chosen set
- ii. the remaining infinitely many eigenvalues do not change

$$\{\lambda_1, \dots, \lambda_p\} \implies \{\mu_1, \dots, \mu_p\}$$

$$\{\lambda_{p+1}, \dots\} \implies \{\lambda_{p+1}, \dots\}$$

**No Change**

Theorem (**Parametric Solution to the Partial Eigenvalue Assignment Problem for a Quadratic Operator Pencil**).

**Part (i) (No-spill-over Part).**

Choose  $\Phi_{kj}$  arbitrarily and define

$$f_{1k} = \sum_{j=1}^p \bar{\Phi}_{kj} M^* v_j$$
$$f_{2k} = \sum_{j=1}^p \bar{\Phi}_{kj} (\bar{\lambda}_j M^* v_j + C^* v_j),$$

**Result:**

Then the **infinite part of the spectrum**  $\{\lambda_{p+1}, \dots\}$  of  $P(\lambda)$  will remain unchanged.

**Part (ii) (Assignment Part).**

- **Solve the Sylvester equation:**

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (v_1, b_1) & \dots & (v_1, b_m) \\ \vdots & & \\ (v_p, b_1) & \dots & (v_p, b_m) \end{pmatrix}.$$

- **Compute**

$$\Phi Z_1 = \Gamma,$$

**Result:**

$$\Omega(P_{c1}(\lambda)) = \{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \dots\}$$

.

## Algorithm. (Parametric Solution to the Partial Eigenvalue Assignment Problem in Distributed Parameter System)

### Inputs:

- (a) The differential operators  $M$ ,  $C$ , and  $K$  of the open-loop pencil  $P(\lambda)$ .
- (b) The  $m$  control functions  $b_1, \dots, b_m$ .
- (c) The set of scalars  $\{\mu_1, \dots, \mu_p\}$ , closed under complex conjugation.
- (d) The self-conjugate subset  $\{\lambda_1, \dots, \lambda_p\}$  of the open-loop spectrum  $\{\lambda_1, \lambda_2, \dots\}$  and the associated eigenfunction set  $\{v_1, \dots, v_p\}$ .

### Outputs:

The feedback functions  $f_1, \dots, f_m$  and  $f_{21}, \dots, f_{2m}$  such that the spectrum of the closed-loop operator pencil is the set  $\{\mu_1, \dots, \mu_p; \lambda_{p+1}, \lambda_{p+2}, \dots\}$ .

## Assumptions:

- The control functions  $b_1, \dots, b_m$  are linearly independent.
- The open-loop quadratic operator pencil  $P(\lambda) = \lambda^2 M + \lambda C + K$  with control functions  $b_1, \dots, b_m$  is partially controllable with respect to the eigenvalues  $\lambda_1, \dots, \lambda_p$ .
- The sets  $\{\lambda_1, \dots, \lambda_p\}$ ,  $\{\lambda_{p+1}, \lambda_{p+1}, \dots\}$ , and  $\{\mu_1, \dots, \mu_p\}$  are disjoint.
- The open-loop operator pencil  $P(\lambda)$  has a discrete spectrum **without finite accumulation points**, every eigenvalue is **Semi-simple**, and the system of eigenfunctions of  $P(\lambda)$  is **two-fold complete**.

(Large Body of Literature on **Spectral Theory of Operators**).

**Step 1.** Form  $\Lambda_1 = \text{diag} (\lambda_1, \dots, \lambda_p)$  and  $\Lambda_{c1} = \text{diag} (\mu_1, \dots, \mu_p)$ .

**Step 2.** Choose arbitrary  $m \times 1$  vectors  $\gamma_1, \dots, \gamma_p$  in such a way that  $\overline{\mu_j} = \mu_k$  implies  $\overline{\gamma_j} = \gamma_k$  and form  $\Gamma = (\gamma_1, \dots, \gamma_p)$ .

**Step 3.** Solve the  $m \times m$  Sylvester equation for  $Z_1$ :

$$\Lambda_1 Z_1 - Z_1 \Lambda_{c1} = \begin{pmatrix} (v_1, b_1) & \dots & (v_1, b_m) \\ \vdots & \ddots & \vdots \\ (v_p, b_1) & \dots & (v_p, b_m) \end{pmatrix} \Gamma.$$

If  $Z_1$  is ill-conditioned, then return to Step 2 and select different  $\lambda_1, \dots, \lambda_p$ .

**Step 4.** Solve the  $m \times m$  linear system:  $\Phi Z_1 = \Gamma$  for  $\Phi = (\phi_{ij})$ .

**Step 5.** If none of the  $\lambda_1, \dots, \lambda_p$  is zero, form for all  $k = 1, \dots, m$

$$f_{1k} = \sum_{j=1}^p \bar{\phi}_{kj} M^* v_j, \text{ and}$$

$$f_{2k} = - \sum_{j=1}^p (\bar{\phi}_{kj} / \bar{\lambda}_j) K^* v_j,$$

otherwise, form for all  $k = 1, \dots, m$ ,

$$f_{1k} = \sum_{j=1}^p \bar{\phi}_{kj} M^* v_j, \text{ and}$$

$$f_{2k} = \sum_{j=1}^p \bar{\phi}_{kj} (\bar{\lambda}_j M^* v_j + C^* v_j).$$

## Distinguished Practical Features

- Only a small finite part of the infinite spectrum (and the associated eigenfunctions) needed to numerically implement the algorithm.
- Mathematical guarantee of **no spill-over**.
- An infinite-dimensional control problem solved using finite-dimensional control and numerically viable finite computational techniques.
- The algorithm is **parametric** in nature. This property can be exploited in designing a **numerically robust feedback control**.

# Case Study With Finite Dimensional Problem

## Vibration of Rotating Axel in a Power Plant

**Mathematical Model:**  $P(\lambda) = \lambda^2 M + \lambda D + K$

- $M = \text{diag} (m_1, m_2, \dots, m_n)$ .
- $D =$  Symmetric tridiagonal
- $K =$  Symmetric tridiagonal

Set  $\gamma_0 = \gamma_n = \kappa_0 = \kappa_n = 0$

$$D = (d_{ij}), \text{ where } d_{ij} = \begin{cases} -\gamma_i & , i + 1 = j \\ \gamma_{i-1} + \delta_i + \gamma_i & , i = j \\ -\gamma_j & , i = j + 1 \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$K = (k_{ij}), \text{ where } k_{ij} = \begin{cases} -\kappa_i & , i + 1 = j \\ \kappa_{i-1} + \kappa_i & , i = j \\ -\kappa_j & , i = j + 1 \\ 0 & , \text{ otherwise} \end{cases}$$

## A Benchmark Example

$$n = 111$$

- **The open-loop Eigenvalues** (222 Eigenvalues)

$$\lambda_1 = -1.3734 \times 10^{-6}$$

**(The Most Unstable Eigenvalue)**

$$R_e(\lambda_j) \leq -0.016267, \quad j = 2, 3, \dots, 422.$$

**(Better Stability Property)**

*The largest contribution to the shape of the transient response is generated by the eigenvectors corresponding to  $\lambda_1$ .*

$\lambda_1 \implies \mu_1 = -0.016$  (**vibration will be suppressed  $10^3$  fold**)

$$x_1 \implies \frac{1}{\sqrt{211}}(1, 1, \dots, 1)^T = y_1.$$

**The control matrix**

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}^T$$

$\Gamma$  = parametric matrix

$$= (-0.51454, -0.85747)^T.$$

## Experimental Results

- $\lambda_1$  was assigned to  $\mu_1$  accurately
- $x_1$  was assigned to  $y_1$  accurately
- 2-Norm difference between the open-loop and closed-loop eigenvalue is about  $1.7 \times 10^{-6}$
- $\|F_1\| < 116$ ,  $\|F_2\| < 22$
- $\frac{\|F_1\|}{\|D\|_2} < 0.57$  and  $\frac{\|F_2\|_2}{\|K\|_2} < 15.10^{-11}$

**(Small Feedback Norms Desirable for Robustness)**

## **Conclusion**

The Vibrations of the rotating turbine axel are suppressed nearly  $10^3$  - fold by using small feedback control forces generated by the Algorithm.

## Finite Element Model Updating Problem:

Given

1. The finite element generated symmetric matrices  $M$ ,  $K$ , and  $D$ :

$$M = M^T > 0, K = K^T \geq 0 \text{ and } D = D^T$$

2. A set of measured eigenvalues  $\{\mu_1, \dots, \mu_m\}$  and the eigenvectors  $\{y_1, \dots, y_m\}$  from a real-life structure.

Find the updated **symmetric updates**  $\tilde{M}$ ,  $\tilde{K}$ , and  $\tilde{D}$  such that

- FEM Eigenvalues  $\longrightarrow$  Measured Eigenvalues
- FEM Eigenvectors  $\longrightarrow$  Measured Eigenvectors
- Remaining Eigenvalues and Eigenvectors  $\equiv$  No Change.

# Finite Element Model Updating (FEMU)

## Finite Element Model

$$M = M^T \geq 0$$

$$K = K^T \geq 0$$

$$D = D^T$$

ANSYS, NASTRAN →

$$\{\lambda_1, \dots, \lambda_p\}$$

**Natural Frequencies**

(Eigenvalues)

and

$$\{x_1, \dots, x_p\}$$

**Mode Shapes**

(Eigenvectors)

## Real-Life Structure

Automobile

Boeing 777

→

$$\{\mu_1, \dots, \mu_p\}$$

**Measured  
Eigenvalues**

and

$$\{y_1, \dots, y_p\}$$

**Measured  
Eigenvectors**

$$\mathbf{FEMU: } M \longrightarrow \tilde{M} = (\tilde{M})^T = M + \Delta M \text{ (Symmetric)}$$

$$K \longrightarrow \tilde{K} = (\tilde{K})^T = K + \Delta K \text{ (Symmetric)}$$

$$D \longrightarrow \tilde{D} = (\tilde{D})^T = D + \Delta D \text{ (Symmetric)}$$

$$\{\lambda_1, \dots, \lambda_p\} \longrightarrow \{\mu_1, \dots, \mu_p\}$$

$$\{x_1, \dots, x_p\} \longrightarrow \{y_1, \dots, y_p\}$$

$$\{\lambda_{p+1}, \dots, \lambda_{2n}\} \longrightarrow \{\lambda_{p+1}, \dots, \lambda_{2n}\} \text{ (No Change)}$$

$$\{x_{p+1}, \dots, x_{2n}\} \longrightarrow \{x_{p+1}, \dots, x_{2n}\} \text{ (No Change)}$$

## Difficulties

- **Finite-Element Models of very High-order.**  
Model Size Needs to be Reduced (**Model Reduction**)
- **Difficult to check no spill-over property computationally or Experimentally.**
- **Incomplete Measured Data.**

(Hard-wire Limitation)

*Analytical Eigenvectors of Full-Length*

*$V_s$*

*Short Measured Eigenvectors.*

*Missing Entries Need to be Supplied.*

- **Complex Data**

*Real Finite Element Data*

*$V_s$*

*Complex Measured Data From Real-life Structures.*

## Challenges

- Problem should be solved without **Model Reduction** or reduction to condensed forms.
- Algorithms should be able to cope up with **Incomplete Measured and Complex Data**
- No spill-over phenomenon to be guaranteed **mathematically**.
- Algorithms should use only the available **small subset of the eigenvalues and eigenvectors** of the quadratic pencil, and the measured data.

## The Current Status of the Problem

- The problem well-studied and still very much active work going on in Vibrating Industries
- Several hundred papers and a book (**Finite Element Model Updating in Structural Dynamics** by M.I. Friswell and J.E. Mottershead, 1995).
- Many Adhoc solutions by Industries (sometimes **Not Based on Sound Mathematical Reasoning**)
- Problem **Not Solved** in desirable way

## Existing Techniques of Model Updating and Drawbacks

- The so-called optimization-based **Direct Methods** deal with **Linear model**:

$$P_i(\lambda) = \lambda M - K$$

rather than the **Quadratic Model**:

$$P_Q(\lambda) = \lambda^2 M + D + K.$$

- **Can not guarantee the no spill-over property.**

*“The updated mass and stiffness matrices have little physical meaning and can to be related to physical changes to the finite-element model in the original model,”* **Friswell and Mottershead.**

## Most Recent Developments

- (B.N. Datta) *Finite Element Model Updating, Eigenstructure Assignment, and Eigenvalue Embedding for Vibrating Systems*, J. Mechanical Vibration and Signal Processing (2003).
- *Ph.D Thesis* of João Carvalho, NIU 2002.

### (The State-of-the-Art-Result on FEMU)

- **Symmetric Eigenvalue Embedding Approach**  
(Carvalho, B.N. Datta, W.W. Lin and J.N. Wang)

**Available at the website:**

[www.math.niu.edu/~dattab](http://www.math.niu.edu/~dattab)

## Finite-Element Model Updating in Undamped Model

(Carvalho '2002).

- The problem **Completely Solved** in the case of Undamped Model
- The difficulties with incomplete measured data resolved in the algorithm itself.

## **PART I (Updating of $K$ with No Spill-over)**

$\Lambda$  = The Finite Element Matrix of Eigenvalues.

$X$  = The Finite Element Matrix of Eigenvectors.

### **Partition**

$$\Lambda = \text{diag}(\Lambda_1, \Lambda_2) :$$

$$\Lambda_1 = \text{diag}\{\lambda_1, \dots, \lambda_p\}$$

$$\Lambda_2 = \text{diag}\{\lambda_{p+1}, \dots, \lambda_{2n}\}$$

$$X = (X_1, X_2) : X_1 = \{x_1, \dots, x_p\}, X_2 = \{x_{p+1}, \dots, x_{2n}\}.$$

## Theorem

Let

$$\tilde{K} = K - MX_1\Phi X_1^T M.$$

Then if  $\Phi$  is a **symmetric matrix**,

(i)  $\tilde{K}$  is a symmetric matrix

and

(ii)  $MX_2\Lambda_2 + \tilde{K}X_2 = 0$

$\implies$  **No Spill-over.**

## PART II (Assignment of Measured Data)

$\Sigma$  = The Matrix of Measured Eigenvalues

$Y_1$  = Matrix of Measured Eigenvectors

**Theorem** Let  $\Phi$  satisfy the Sylvester matrix equation:

$$(Y_1^T M X_1) \Phi (Y_1^T M X_1) = Y_1 M Y_1 \Sigma + Y_1^T K Y_1.$$

• Then  $\Phi$  is **symmetric**

•  $\Omega(\lambda^2 M + \tilde{K}) = \{ \text{Measured eigenvalues; } \lambda_{p+1} \dots, \lambda_{2n} \}$

• Eigenvectors of  $(\lambda^2 M + \tilde{K}) : \{ \text{Measured eigenvectors; } x_{p+1} \dots, x_{2n} \}$ .

**Notes:**  $Y_1$  = Measured Eigenvector Matrix

= **Not Completely Known**

$$= \begin{pmatrix} Y_{11} \longleftarrow \text{Known} \\ Y_{12} \longleftarrow \text{Unknown} \end{pmatrix}$$

- The unknown part is computed appropriately by the Algorithm.

*Model Updating of an Undamped Symmetric Positive Semidefinite Model Using Incomplete Measured Data*

**Input:** The symmetric matrices  $M, K \in \mathbb{R}^{n \times n}$ ; the set of  $m$  analytical frequencies and mode shapes to be updated; the complete set of  $m$  measured frequencies and model shapes from the vibration test.

**Output:** Updated stiffness matrix  $\tilde{K}$ .

**Assumption:**  $M = M^T \geq 0$  and  $K = K^T \geq 0$ .

**Step 1:** Form the matrices  $\Sigma_1^2 \in \mathbb{R}^{m \times m}$  and  $Y_{11} \in \mathbb{R}^{m \times m}$  from the available data. form the corresponding matrices  $\Lambda_1^2 \in \mathbb{R}^{m \times m}$  and  $X_1 \in \mathbb{R}^{n \times m}$ .

**Step 2:** Compute the matrices  $U_1 \in \mathbb{R}^{n \times m}$ ,  $U_2 \in \mathbb{R}^{n \times (n-m)}$ , and  $Z \in \mathbb{R}^{m \times m}$  from the QR factorization:

$$MX_1 = [U_1 \ U_2] \begin{bmatrix} Z \\ 0 \end{bmatrix}$$

**Step 3:** Partition  $M = [M_1 \ M_2]$ ,  $K = [K_1 \ K_2]$  where  $M_1, K_1 \in \mathbb{R}^{n \times m}$ .

**Step 4:** Solve the following matrix equation to obtain  $Y_{12} \in \mathbb{R}^{(n-m) \times m}$ .

$$U_2^T M_2 Y_{12} \Sigma + U_2^T K_2 Y_{12} = -U_2^T [K_1 Y_{11} + M_1 Y_{11} \Sigma]$$

and form the matrix

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}.$$

## Theorem on Symmetry Preserving Partial Eigenvalue Assignment

Let  $(\lambda_1, y_1)$  be an unwanted real isolated eigenpair of  $P(\lambda) = \lambda^2 M + \lambda D + K$  with  $y_1^T K y_1 = 1$ . Let  $\lambda_1$  be reassigned to  $\mu_1$ . Define  $\theta_1 = y_1^T M y_1$  and assume that  $1 - \lambda_1 \mu_1 \theta_1 \neq 0$  and  $1 - \lambda_1^2 \theta_1 \neq 0$ .

- $(\lambda_1, Y_1)$  - An Unwanted Isolated Real Eigenpair
- $\theta_1 = y_1^T M Y_1$
- $\epsilon = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \theta_1}$
- Updated model  $P_U(\lambda) = \lambda^2 M_U + \lambda D_U + K_U$

is such

$$M_U = M - \epsilon_1 \lambda_1 M y_1 y_1^T M$$

$$D_U = D + \epsilon_1 (M y_1 y_1^T K + K y_1 y_1^T M)$$

$$K_U = K - \frac{\epsilon_1}{\lambda_1} K y_1 y_1^T K$$

that

- i. The eigenvalues of  $P_U(\lambda)$  the same as those of  $P(\lambda)$  except that  $\lambda_1$  replaced by  $\mu_1$ .
- ii.  $y_1$  also an eigenvector of  $P_U(\lambda)$  corresponding to the embedded eigenvalue  $\mu_1$ .
- iii. If  $(\lambda_2, y_2)$  an eigenpair of  $P(\lambda)$ , where  $\lambda_2 \neq \lambda_1$ , then  $(\lambda_2, y_2)$  also an eigenpair of  $P_U(\lambda)$ .

## Conclusions

- Some very interesting (but **very difficult**) **Structured Inverse Eigenvalue Problems** arising in practical Industrial Applications.
- **Real-life applicable** and **mathematically sound** solutions.
- Many existing industrial techniques are **ad-hoc** in nature. *Not much consideration for mathematical difficulties and challenges.*
- Very often *lacks strong mathematics foundations.*

- Industries in **Japan** and **Germany** take more **mathematical approach to industrial problems**.
- Need people with **industrial aptitude and interdisciplinary training** blending *Linear Algebra, Numerical Linear Algebra, and Scientific Computing* with areas of engineering such as *Mechanical and Electrical Engineering*. **Such expertise are rare.**
- Curricular in both **Engineering, Mathematics and Computer Science** need to be re-looked into for opportunities for **interdisciplinary courses**.
- Many **engineering text books** need to be rewritten incorporating recent developments in **matrix computations, scientific computing and mathematical software**.