

**A New Hybrid Method for Finding
Eigenpairs of a Symmetric Quadratic
Eigenvalue Problem in an Interval**

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Over View

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- II. Parametrized Newton Method to Symmetric Quadratic Eigenvalue Problem
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I. Introduction

The symmetric quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)u = 0,$$

where M , C , and K are given $n \times n$ matrices and (λ, u) is an eigenpair

Some Applications:

- **Vibration Analysis of Structural Systems**
Undamped structural eigenvalueproblem in industrial engineering

$$(\lambda^2 Mu + Ku) = 0,$$

In practice systems are subject to damping. Modern structures lead to non-classically damped eigenvalueproblems

$$\lambda^2 Mu + \lambda Cu + Ku = 0.$$

- **Constrained least squares problem**

$$\min_{x^T x = \alpha^2} \{x^T A x - 2b^T x\}$$

where $A = A^T \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$.

The solution is $x = (A - \lambda I)^{-1}b$, where λ is the smallest eigenvalue of

$$(\lambda^2 I + 2\lambda A + (A^2 - \alpha^{-2}bb^T))y = 0,$$

- **Eigenvalue assignment problem for quadratic matrix pencil**

A Brief Review of the Existing Methods:

The Quadratic eigenvalue problem of the form $(\lambda^2 M + \lambda C + K)x = 0$ is usually solved in two stages.

Stage I. Transform QEP into equivalent generalized Eigenvalue problem

$$Ay = \lambda By$$

where

$$A = \begin{pmatrix} -C & -K \\ I & O \end{pmatrix}$$

and

$$B = \begin{pmatrix} M & O \\ O & I \end{pmatrix}$$

(M is a non singular $n \times n$ matrix).

Stage II. Computing the Eigenpairs.

Then find all the eigenpairs by using QZ algorithm if n is small and dense, or apply Krylov-subspace-based methods to find few eigenvalues of the QEP .

Ib. OTHER METHODS

- Jacobi-Davidson Method for QEP :

In this method, QEP is not transformed into linear form; instead this method projects QEP directly onto a properly chosen low-dimensional subspace where QEP can be solved directly by a standard dense matrix technique.

- A recent second-order Arnoldi Method for the solution of the QEP (Bai and Su (2005)).

Ic. Drawback of the Existing Methods:

- The coefficient matrices M, C, K are symmetric positive definite but by the process of linearization, the transformed generalized eigenvalue problem becomes nonsymmetric, or symmetric indefinite. Subsequently the essential spectral properties of QEP are not guaranteed to be preserved.
- Also the generalized eigenvalue problem is twice the dimension of the original QEP.
- Successes of the Jacobi-Davidson Method strongly depends how to choose the initial eigenpair. Also this method targets one eigenpair at a time with local convergence verses Krylov Subspace in which a group of eigenvalues is approximated with global convergence.
- Second-order Arnoldi method doesnot give good approximation of eigenpair closer to the lower part of the spectrum. Also with this method it is not easy to find eigenvalues in a specific interval.

II. The Parametrized Newton Method to the Symmetric Quadratic Eigenvalue Problem

Define function: $f : R^{n+1} \rightarrow R^{n+1}$

$$f(\lambda, u) = \begin{pmatrix} Q(\lambda)u \\ u^T u - 1 \end{pmatrix},$$

where $Q(\lambda) = \lambda^2 M + \lambda C + K$,
 M , C , and $K \in R^{n \times n}$ are symmetric positive definite matrices and $u \in R^n$ and $\lambda \in R$.

Jacobian matrix J_f of f which can be calculated as

$$J_f(\lambda, u) = \begin{pmatrix} Q(\lambda) & Q'(\lambda)u \\ 2u^T & 0 \end{pmatrix},$$

where $Q'(\lambda) = 2\lambda M + C$ is the derivative of the matrix polynomial $Q(\lambda)$.

***The Parametrized Newton Iterations for
Symmetric Quadratic Pencil are:***

$$\begin{pmatrix} x_{i+1} \\ \alpha_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ \alpha_i \end{pmatrix} - \begin{pmatrix} Q(\alpha_i) & Q'(\alpha_i)x_i \\ 2x_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} Q(\alpha_i)x_i \\ x_i^T x_i - 1 \end{pmatrix}.$$

Now choose a parameter $t > 0$ and assume $\alpha_i \neq 0$ so that the method takes the form:

$$\begin{pmatrix} Q(\alpha_i) & Q'(\alpha_i)x_i \\ 2x_i^T & 0 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ \alpha_{i+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & Q'(\alpha_i)x_i \\ x_i^T & \frac{1}{\alpha_i} \end{pmatrix} \begin{pmatrix} x_i \\ \alpha_i \end{pmatrix}.$$

*Parametrized Newton's iteration for $Q(\lambda)$
now takes the form:*

$$x_{i+1} = \frac{1}{\hat{\beta}_i} Q^{-1}(\alpha_i) \cdot Q'(\alpha_i) x_i \quad (1)$$

$$\alpha_{i+1} = \alpha_i - \frac{r_i}{\hat{\beta}_i} s \quad (2)$$

$$\hat{\beta}_i = \|Q^{-1}(\alpha_i) Q'(\alpha_i) x_i\|$$

$$\beta_i = x_i^T Q^{-1}(\alpha_i) Q'(\alpha_i) x_i$$

$$r_i = \frac{\beta_i}{\hat{\beta}_i}$$

where we choose the value s that gives the minimum residual

III. Convergence Criteria:

we define the the residual at the $(i+1)^{th}$ step of Parametrized Newton's Method by

$$Res_{i+1} = Q(\alpha_{i+1})x_{i+1}. \quad (3)$$

Using Parametrized Newton's Iteration, we can then show that:

$$\begin{aligned} \|Res_{i+1}\|^2 &= \frac{1}{\hat{\beta}_i^2} [y_i^T y_i - 2\frac{r_i s}{\hat{\beta}_i} y_i^T p_i \\ &\quad + \frac{r_i^2 s^2}{\hat{\beta}_i^2} (2y_i^T z_i + p_i^T p_i) - 2\frac{r_i^3 s^3}{\hat{\beta}_i^3} p_i^T z_i + \frac{r_i^4 s^4}{\hat{\beta}_i^4} z_i^T z_i]. \end{aligned} \quad (4)$$

That is,

$$\begin{aligned} \|Res_{i+1}\|^2 &\leq \frac{1}{\hat{\beta}_i^2} [\|y_i\|^2 - 2\frac{r_i s}{\hat{\beta}_i} y_i^T p_i + 2\frac{r_i^2 s^2}{\hat{\beta}_i^2} \|y_i\| \|z_i\| \\ &\quad + \frac{r_i^2 s^2}{\hat{\beta}_i^2} \|p_i\|^2 - 2\frac{r_i^3 s^3}{\hat{\beta}_i^3} p_i^T z_i + \frac{r_i^4 s^4}{\hat{\beta}_i^4} \|z_i\|^2] \end{aligned}$$

where $\hat{\beta}_i^2 = \|z_i\|^2$, $y_i = Q'(\alpha_i)x_i$,

$$z_i = Q^{-1}(\alpha_i)Q'(\alpha_i)x_i, \quad p_i = Q'(\alpha_i)z_i,$$

IV. The New Hybrid Method:

- 1) a. we choose 3 sets of random eigenpairs $\begin{pmatrix} \alpha_i \\ v_i \end{pmatrix}$
where $\alpha_i \in [a, b]$, and the vectors $v_i, i = 1, 2, 3$
are orthogonal to each other.

b. Use 3-iterations of Parametrized Newton's method
- 2) Use these three eigenvectors to run the Jacobi-Davidson
method.

***New Hybrid Algorithm to Compute an
Eigenpair of a Symmetric Quadratic Pencil
in an Interval:***

Define:

$$Q(\lambda) = \lambda^2 M + \lambda C + K; \quad Q'(\lambda) = 2\lambda M + C$$

Where M, C, K are given

Let $[a, b]$ be the given interval.

Algorithm:

INPUT:

- The matrix $M = I \in R^{n \times n}$, C , and $K \in R^{n \times n}$ symmetric positive definite matrices.
- Three real numbers $\alpha_i, i=1, 2, 3$ as initial approximations of an eigenvalues inside the interval $[a b]$.
- An orthonormal matrix $V = (v_1, v_2, v_3)$.

OUTPUT:

An approximate eigenpair of the QEP in the interval $[a b]$.

Step 1: Find three approximate eigenpairs (α_i, v_i) using the Parametrized Newton's method described in Section 2 (run only maximum of three iterations).

Step 2: Apply the Jacobi-Davidson method with the eigenvectors obtained in the Step 1 and a shift α choosing it as one of the eigenvalues $\alpha_i, i = 1, 2, 3$ appropriately (depending upon the location of the eigenvalue sought).

Step 3: Check if norm of the residual defined in equation (3) is less than a given tolerance. If so Stop. Otherwise, expand the search space V and return to Step 2.

V. Results of Numerical Experiments:

M is identity matrix,

C , K are arbitrary symmetric positive definite matrices of order 500 and 800

Maximum number of iterations = 20

and tolerance = 10^{-4} .

Example 1

Matrix size(n) = 500, Interval [41.5 43.5]

The approximate initial eigenvalue determined by the Parametrized Newton method = 42.0307903.

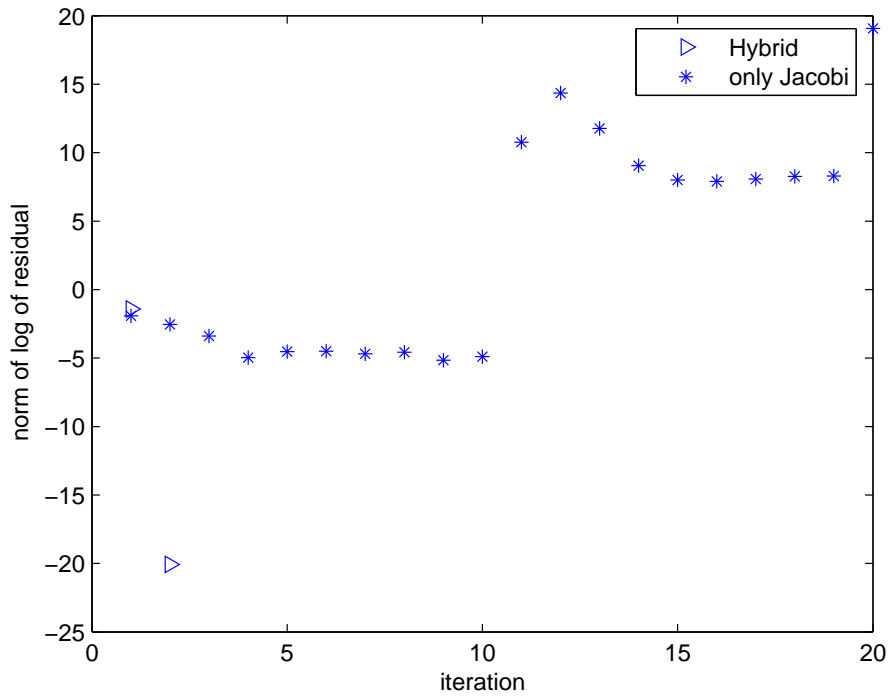
Exact eigenvalues in interval [41.5 43.5] are:

41.59431775969728, 41.62015325607620, 41.93211289355256, 43.00123828531239.

TABLE 1: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 1

Methods	Residual	Iteration	Eigenvalue
Hybrid	$1.9e^{-9}$	2	41.93211
JD	No convergence	20	

Figure 1: Norm of log of Residual *verses* Iteration



Example 2

Matrix size(n) = 800, Interval [70 73]

The approximate initial eigenvalue determined by the Parametrized Newton method = 71.063630762458.

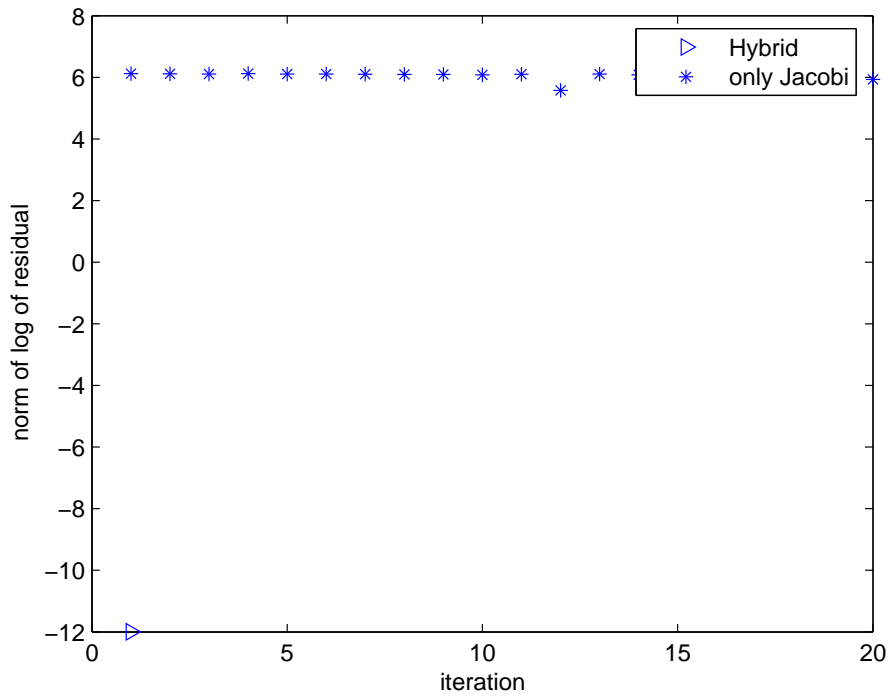
Exact eigenvalues in interval [70 73] are:

71.48881531878268, 72.89726841255406.

TABLE 2: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 2

Methods	Residual	Iteration	Eigenvalue
Hybrid	$6.61e^{-6}$	1	71.4888153978
JD	No convergence	20	

Figure 2: Norm of log of Residual *verses* Iteration



Example 3

Matrix size(n) = 800, Interval [60 62]

The approximate initial eigenvalue determined by the Parametrized Newton method = 60.995819719.

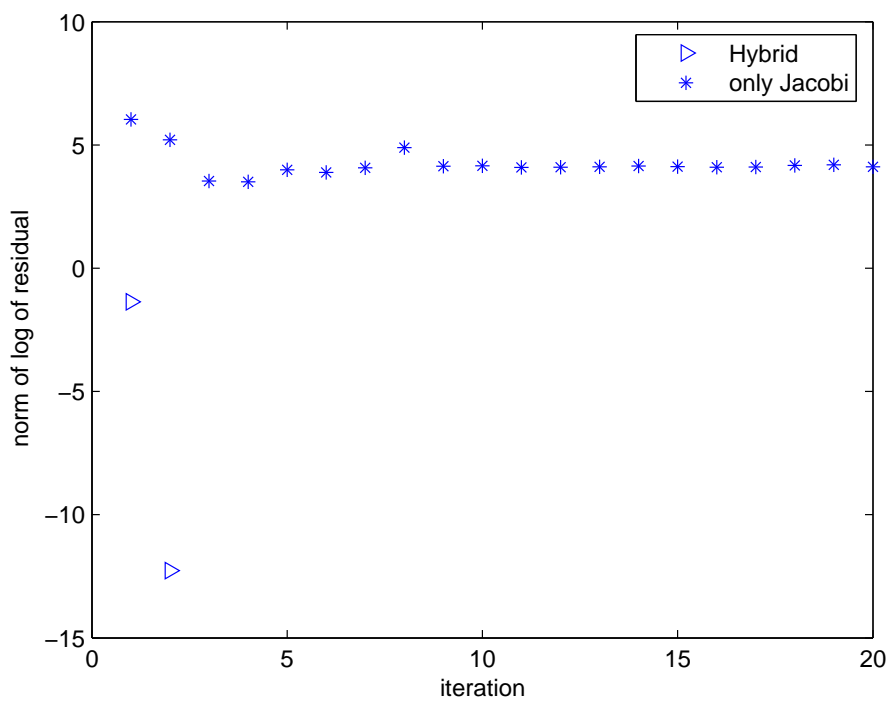
Exact eigenvalues in interval [60 62] are:

60.38940860998, 60.95958197, 61.344186171,
61.796883006, 61.937923103.

TABLE 3: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 3

Methods	Residual	Iteration	Eigenvalue
Hybrid	$5.0e^{-6}$	2	60.38940860998
JD	No convergence	20	

Figure 3: Norm of log of Residual *verses* Iteration



Conclusion:

- Numerical experimental results show that the hybrid method converges faster than the Jacobi-Davidson method alone for symmetric QEP; indeed, in some cases when the Jacobi-Davidson method did not converge at all, the new method worked quite well
- The method is useful for many practical scientific and engineering applications.
- Studies on how to choose the parameter s properly to guarantee or accelerate the convergence is currently underway and will be reported in a future paper.

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THANK YOU