Some Solutions for Math 480 Week #13

Page 373 #1: Show that $\sum_{d\le x}\mu(d)[x/d]=1$ for all $x\ge 1$. Deduce that $|\sum_{d\le x}\mu(d)/d|\le 1$ for all $x\ge 1$.

Write $[x/d]=\sum_{m\le x/d}1$. Then setting $n=dm$ below gives

$$\begin{aligned}\sum_{d\le x}\mu(d)[x/d] &= \sum_{d\le x}\mu(d)\sum_{m\le x/d}1\\ &= \sum_{n\le x}1\sum_{d|n}\mu(d)\\ &= 1,\end{aligned}$$

since $\sum_{d|n}\mu(d)=0$ unless $n=1$, in which case it is 1.

Now write $[x/d]=(x/d)-\{x/d\}$. Note that $0\le\{x/d\}<1$ always. Then

$$1=\sum_{d\le x}\mu(d)[x/d]=x\sum_{d\le x}\frac{\mu(d)}{d}-\sum_{d\le x}\mu(d)\{x/d\}$$

and

$$\frac{1}{x}+\frac{1}{x}\sum_{d\le x}\mu(d)\{x/d\}=\sum_{d\le x}\frac{\mu(d)}{d}.$$

Suppose first that x is a positive integer. Then $\{x/1\}=0$, so we have

$$\begin{aligned}\left|\sum_{d\le x}\frac{\mu(d)}{d}\right| &= \left|\frac{1}{x}+\frac{1}{x}\sum_{d\le x}\mu(d)\{x/d\}\right|\\ &= \left|\frac{1}{x}+\frac{1}{x}\sum_{2\le d\le x}\mu(d)\{x/d\}\right|\\ &\le \frac{1}{x}+\frac{1}{x}\sum_{2\le d\le x}|\mu(d)\{x/d\}|\\ &\le \frac{1}{x}+\frac{1}{x}\sum_{2\le d\le x}\{x/d\}\\ &\le \frac{1}{x}+\frac{x-1}{x}\\ &= 1.\end{aligned}$$

If x is not an integer, simply use

$$\sum_{d\le x}\frac{\mu(d)}{d}=\sum_{d\le[x]}\frac{\mu(d)}{d}$$

and what we just proved.

#2: Show that $\Lambda(n) = -\sum_{d|n} \mu(d) \log d$ for every positive integer n.

Since the equation is obviously true if $n = 1$, assume $n > 1$. By Theorem 8.1, $\sum_{d|n} \Lambda(d) = \log n$. By Möbius inversion (Theorem 4.8) and using $\log(n/d) = \log n - \log d$, we get

$$\begin{aligned}\Lambda(n) &= \sum_{d|n} \mu(d) \log(n/d) \\ &= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d \\ &= -\sum_{d|n} \mu(d) \log d.\end{aligned}$$

Here we used $\sum_{d|n} \mu(d) = 0$, since $n \neq 1$.

#4: Show that $2^x < \prod_{p \leq x} p < (13/4)^x$ for all sufficiently large x.

We have

$$\log\left(\prod_{p \leq x} p\right) = \sum_{p \leq x} \log p = \theta(x).$$

We showed in class that $\theta(x) \leq \psi(x)$. Using the upper bound for $\psi(x)$ we proved in class and a calculator gives

$$\psi(x) \leq x\left(\frac{\log 2}{3} + \frac{\log 3}{2}\right)\frac{3}{2} + 5\left(\log x + 1\right)^2 < 1.171x + 5\left(\log x + 1\right)^2.$$

Moreover, you can verify via calculus that

$$\lim_{x \to \infty} \frac{(\log x + 1)^2}{x} = 0.$$

Thus, for x sufficiently large, we have $.007x \geq 5(\log x + 1)^2$. Putting all this together gives

$$\theta(x) \leq 1.171x + .007x = 1.178x, \qquad \prod_{p \leq x} p < e^{1.178x} = \left(e^{1.178}\right)^x.$$

Again using a calculator, you can verify that $e^{1.178} < 13/4$. This proves the upper bound.

We showed in class that $\theta(x) \geq \psi(x) - \log(x)\theta(x^{1/2}) \geq \psi(x) - \log x \psi(x^{1/2})$. To make things look nicer, set $c = \left(\frac{\log 2}{3} + \frac{\log 3}{2}\right)$. Using the upper and lower bounds for $\psi(x)$ we proved in class gives

$$\begin{aligned}\psi(x) - \log x \psi(x^{1/2}) &\geq cx - 5(\log x + 1) - 1.5cx^{1/2} \log x - 5(\log x^{1/2} + 1)^2 \log x \\ &\geq cx - 1.5cx^{1/2} \log x - 10(\log x + 1)^3.\end{aligned}$$

Similar to the limit above, we have

$$\lim_{x\to\infty} \frac{(\log x + 1)^3}{x} = 0.$$

Thus, for x sufficiently large, $(\log x + 1)^3 < 10^{-9}x$. This also implies that $\log x < 10^{-3}x^{1/3}$. For such sufficiently large x we have

$$\begin{aligned}\theta(x) \geq \psi(x) - \log x\psi(x^{1/2}) &\geq cx - 1.5cx^{1/2}\log x - 10(\log x + 1)^3\\ &> cx - 1.5cx^{1/2}10^{-3}x^{1/3} - 10^{-8}x\\ &> cx - 10^{-2}x\\ &> .79x.\end{aligned}$$

This estimate gives

$$\prod_{p\leq x} p > e^{.79x} = (e^{.79})^x > 2^x$$

for all sufficiently large x.

In hindsight, we see that our estimates for $\psi(x)$ allow us to show that $\prod_{p\leq x} p < (e^{c_1})^x$ for all sufficiently large x as long as $c_1 > 1.5c$, and that $\prod_{p\leq x} p > (e^{c_2})^x$ for all sufficiently large x as long as $c_2 < c$.

#8: Prove that $n! = m^k$ is impossible in integers $k > 1,\ m > 1,\ n > 1$.

This is easily verified when $n = 2$, so assume that $n \geq 3$. Since $n/2 > 1$, Bertrand's Postulate says there is at least one prime p with $n > p > n/2$. If $n \geq x \geq n/2$, then $p > x/2$, so either $p = x$ or p doesn't divide x. If $x < n/2$, then $p > x$ and p doesn't divide x. Thus, the only number x between 1 and n which is divisible by p is p itself. This shows that $n! = p \cdot r$ for some integer r not divisible by p. But by the Fundamental Theorem of Arithmetic, any prime q dividing m^k must divide m, so that q^k divides m^k. Since p^k doesn't divide $n!$ for any $k > 1$, $n! \neq m^k$.